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THE DOCTRINE OF ANNUITIES and REVERSIONS,

DEDUCED FROM
GENERAL and EVIDENT PRINCIPLES:
WITH USEFUL
TABLES,
SHEWING THE VALUES OF
SINGLE and JOINT LIVES, &c.
AT DIFFERENT
RATES OF INTEREST.

TO WHICH IS ADDED,
A METHOD of inveſtigating the Value of ANNUITIES by Approximation, without the help of TABLES.

The Whole explained in a plain and ſimple Manner,
and illuſtrated by great Variety of Examples.

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THE

PREFACE.

THE ſubject upon which this little Tract is founded, as to its uſefulneſs, needs no apology; and I have endeavoured to put it in ſuch a light, as may fully anſwer the ends and expectation of the reader.

For, in the firſt place, I have given a very exact table for eſtimating the probability of life, deduced from 10 *years obſervations on the bills of mortality of the city of* London, *whereupon the ſucceeding calculations are grounded. Then, after ſhewing how to compute the values of ſingle lives, I lay down a*

Lemma, for the ſake of thoſe unacquainted with the principles of chances, by help whereof the moſt intricate Problems in the ſubject are reſolved. Next the values of annuities upon more lives than one come to be conſidered; as, firſt, for any number of joint *lives; ſecondly, for the longeſt of any number of lives; thirdly, for lives, where the annuity ceaſes upon the extinction of any aſſigned number of them. Then I proceed to determine the values of reverſions, firſt, for the longeſt of any number of lives, after the longeſt of any number of other lives; ſecondly, for any number of* joint *lives, after any number of* joint *lives; thirdly, for any number of* joint *lives, after the longeſt of any number of lives; fourthly, for the longeſt of any number of lives, after any number of* joint *lives. Then, from the theorems before laid down, is given a ſet of tables for the valuation of annuities, upon one, two, or three lives, according to ſeveral rates of intereſt; the uſes of which are explained in ſuch a manner, as to be underſtood by all who know but common arithmetic.*

Next is ſhewn how to determine the values of Succeſſive Lives, where the firſt poſſeſſor has a right, at his deceaſe, to nominate his ſucceſſor, and his ſucceſſor a next ſucceſſor, and ſo on. Then the value that ought to be paid for renewing of leaſes upon any number of lives, together with the loſs or gain of the purchaſer in renewing for any aſſigned ſum is conſidered; as alſo how much the rent roll of an eſtate ought to be increaſed upon account of ſuch renewals.

Then is given a method for finding the values of Reverſions, when the expectation depends on the chance of one particular life in poſſeſſion ſurviving the reſt.

Laſtly, are laid down ſome eaſy practical Rules for approximating the values of lives, without the help of tables, both according to the London *and* Breſlaw *obſervations, which anſwer very near the truth, but that in p.* 121 *I find will be ſtill more exact, if inſtead of* 1.9 *the number therein mentioned, the number* 2 *be made uſe of, when the propoſed age is greater than thirty years; for then the* *rule may be extended from eight to eighty years of age, and yet the error ſcarce ever amount to one tenth of a year's purchaſe.*

What, I apprehend, may beſt recommend this performance, is the general, yet familiar manner in which the ſubject is treated, there not being a Solution throughout the whole work, except thoſe relating to the uſe of the tables, that is not univerſal, according to any table of obſervations or degree of probability of life whatſoever; and yet the concluſions and practical rules deduced therefrom, are, for the moſt part altogether as ſimple, as could be derived from any hypotheſis. I mention hypotheſes, becauſe ſome authors on this ſubject, without troubling themſelves or their readers about obſervations, &c. *have taken upon them to preſcribe methods of their own, that have neither foundation in experience nor in reaſon.*

Having taken ſome notice of the defects of others, it may not be improper to endeavour to obviate an objection that may be made to the enſuing work.

It is poſſible that the great difference which there is, in one part of life, between the value of an annuity according to the Breſlau *obſervations, and the following tables, may tempt ſome (eſpecially thoſe whoſe intereſt it is) to queſtion the exactneſs of the tables, or the obſervations whereon they were grounded; they may affirm that* London *conſiſts of too flux a body to admit of any certain meaſure for the probability of life, and that the accounts publiſhed by the company of pariſh-clerks, are not to be depended on. But to this I anſwer, that tho' the continual reſort of people from all parts, cauſes indeed a great increaſe in the bills of mortality, it will no ways influence the values of the annuities thence deduced, if the numbers of perſons coming up to town at all ages, be proportional to the whole numbers of the living of the ſame ages; and tho' this ſuppoſition is not exactly true in ſmall ages, yet as experience ſhews it to be more nearly ſo in greater, and as the number of perſons that come to live in town after* 25 *or* 30 *years of age, is inconſiderable with reſpect to the whole body of inhabitants, it is evident, that the* *values given in the tables for all ages, not leſs than 25 or 30 years, can be but little effected from the cauſe abovementioned. It is true, the values for younger lives, have not quite ſo good a foundation; but, I preſume the method I have had recourſe to upon this occaſion is ſuch, as is not liable to any reaſonable objections; and, as to the difference that may ariſe from any uncertainty or error in the accounts of the pariſh-clerks, it can be but very little, becauſe if the age happens to be given in a little too high one time, there is the ſame chance of its being put down a much too low another.*

N. B. Since the publication of this Work in 1742, the author has further proſecuted this ſubject by way of Supplement to it, in part VI. of his SELECT EXERCISES, to which the reader is referred.

OF THE VALUATION OF ANNUITIES upon LIVES.

THE value of an annuity for life, depends upon the intereſt which money bears, and the probability of the life continuing a longer or ſhorter time; the former of which is generally ſettled by law, but the latter muſt be determined from obſervations.

Of all that has been hitherto offered for eſtimating the probability of the duration of life, nothing ſeems deduced with greater judgment and exactneſs, than the tables publiſhed by Dr. *Halley*, and Mr. *Smart*, for this purpoſe; which, nevertheleſs, are both liable to conſiderable objections.

The Doctor's Table, being grounded on obſervations made at *Breſlau*, a place where the generality of people live to a greater age than at *London* (as appears by comparing the bills of mortality here, with thoſe obſervations) can be no juſt meaſure of the probability of life in this place; and as to that of Mr. *Smart*, tho' it is indeed free from this objection, and founded on a very large number of obſervations, yet the great and continual afflux of people from all parts up to town, renders the deductions from thoſe obſervations conſiderably different, in one part of life, from what they would otherwiſe be; and this Mr. *Smart* ſeems not, in his table, to have conſidered, or made any allowance for.

For theſe reaſons, tho' I had determined to depend on, and make uſe of, this laſt gentleman's obſervations, in the enſuing pages (as, undoubtedly, the beſt for the city of *London*, and parts adjacent) yet have I deemed it neceſſary to make ſome alterations, in the table of the probability of life, from thence derived.

In doing this, I have ſuppoſed the number of perſons coming to live in town, after 25 years of age, to be inconſiderable, with reſpect to the whole number of inhabitants; and therefore the probabilities of life, for all ages above 25 years, the ſame as this author has made them; but then have increaſed the numbers of the living, correſponding to all ages below 25; ſo that they may, as near as poſſible, be in the ſame proportion one to another, as they would be, were they to be deduced from obſervations on the mortality of thoſe perſons *only*, that are born within the bills. Which was done, by comparing together the number of chriſtenings and burials, and obſerving, by help of Dr. *Halley's* table, the proportion which there is between the degrees of mortality at *London* and *Breſlau*, in the other parts of life, where the ages are greater than 25. I ſhall here ſubjoin the table, altered as above, and then proceed immediately to the uſes thereof.

A TABLE *ſhewing the Proba-*

Note. The Numbers marked × are ſuppoſed to die called the Decrements of Life.

Nº. of Perſons.	Ages curt.	Nº. of Perſons.	Ages curt.	Nº. of Perſons.	Ages curt.	Nº. of Perſons.	Ages curt.
1280	born	524	10	462	20	385	30
410 ×		7 ×		7 ×		9 ×	
870	1	517	11	455	21	376	31
170 ×		7 ×		7 ×		9 ×	
700	2	510	12	448	22	367	32
65 ×		6 ×		7 ×		9 ×	
635	3	504	13	441	23	358	33
35 ×		6 ×		7 ×		9 ×	
600	4	498	14	434	24	349	34
20 ×		6 ×		8 ×		9 ×	
580	5	492	15	426	25	340	35
16 ×		6 ×		8 ×		9 ×	
564	6	486	16	418	26	331	36
13 ×		6 ×		8 ×		9 ×	
551	7	480	17	410	27	322	37
10 ×		6 ×		8 ×		9 ×	
541	8	474	18	402	28	313	38
9 ×		6 ×		8 ×		9 ×	
532	9	468	19	394	29	304	39
8 ×		6 ×		9 ×		10 ×	
524	10	462	20	385	30	294	40

bilities of LIFE, *from Obſervations.*

off yearly, and are what, in the ſucceeding Pages, are

Nº. of Perſons.	Ages cur^{t}.	Nº. of Perſons.	Ages cur^{t}.	Nº. of Perſons.	Ages cur^{t}.	Nº. of Perſons.	Ages cur^{t}.
294	40	204	50	130	60	69	70
10 ×		8 ×		7 ×		5 ×	
284	41	196	51	123	61	64	71
10 ×		8 ×		6 ×		5 ×	
274	42	188	52	117	62	59	72
10 ×		8 ×		6 ×		5 ×	
264	43	180	53	111	63	54	73
9 ×		8 ×		6 ×		5 ×	
255	44	172	54	105	64	49	74
9 ×		7 ×		6 ×		4 ×	
246	45	165	55	99	65	45	75
9 ×		7 ×		6 ×		4 ×	
237	46	158	56	93	66	41	76
9 ×		7 ×		6 ×		3 ×	
228	47	151	57	87	67	38	77
8 ×		7 ×		6 ×		3 ×	
220	48	144	58	81	68	35	78
8 ×		7 ×		6 ×		3 ×	
212	49	137	59	75	69	32	79
8 ×		7 ×		6 ×		3 ×	
204	50	130	60	69	70	29	80

Now, in order to ſhew the uſe of the foregoing table by an example, let it be required to find the probability, that a perſon of thirty-ſix, lives 30 years longer, or attains to the age of 66 years, look in the table againſt 36 years and 66 years, and correſponding thereto, you will find the numbers 331 and 93 reſpectively; ſhewing, that out of 331 perſons living of 36 years of age, only 93 of them arrive to the age of 66: therefore, ſeeing the whole number of perſons living at the beginning of this term, is to the number remaining alive at the end of it, in the ratio of 331 to 93; the number of chances that a perſon of 36 years of age has to live 30 years longer, will be to the number of all the chances, that he has both to live beyond, and die within 30 years, in the ſame ratio of 331 to 93; and therefore $\frac{93}{331}$ is the meaſure of the probability required; the probability of the happening of any event, being always to be conſidered as the ratio of the chances which that event has to happen, to all the chances which it has both to happen and fail.

This being underſtood, ſuppoſe it were now required to find the value of an annuity of 100*l.* for a life of 20 years of age, intereſt at 4 *per cent.*

Becauſe the preſent value of 100*l.* due at the end of one year (diſcount being allowed) is 96.15, it is plain, that ſo much would be the value of the firſt year's rent, was the purchaſer ſure to receive it; but the probability of his living one year, appearing from the table to be only $\frac{455}{462}$, the aforeſaid ſum 96.15, in order to make a juſt deduction out of it, for the contingency of his dying before the end of one year, ought to be diminiſh'd in the ratio of 462 to 455, or multiply'd by $\frac{455}{462}$, which will reduce it to 94.70, equal to the true value of the firſt year's rent. After the ſame manner may the value of the ſecond year's rent be calculated; for ſince the probability of receiving this rent, or living two years is $\frac{448}{462}$, let this be multiplied into 92.45, the preſent value of 100*l.* to be received at the end of two years, and the product 89.65, will be the true value of the ſecond year's rent.

And by a like way of proceeding, the values of the 3d, 4th, 5th, *&c.* year's rents, to the utmoſt extent of life, may be determined; and the ſum of all theſe will be the required value of the annuity; which will be found to come out 1480*l.* very near.

From the ſame method of proceeding, the value of an annuity for any other life may be determined: and tho' the operations, requiſite to this effect, are very numerous, yet, that being once computed, the value of the next younger life may from thence be eaſily derived; for if to the given value you add one year's purchaſe, and multiply the ſum, diſcounted for one year, by the probability of the youngeſt life continuing one year, the product will be the value required (as will appear from Corollary VII. of the ſucceeding Problem.) Suppoſe, for example, that the value of a life of nineteen were required from the value of a life of twenty, as above computed; then, the value of the given life, increaſed by one year's purchaſe, being 1580*l.* the ſame diſcounted for one year, at 4 *per cent.* will be 1519.2, which multiply'd by $\frac{462}{468}$, the probability of a life of nineteen continuing one year, gives 1499.8, for the required value of an annuity upon this life.

Having ſhewn the manner of eſtimating the value of an annuity for any ſingle life, and laid down a ready method of computing tables for ſuch lives, according to any propoſed rate of intereſt, by deducing each value from that of the next older life, it remains next to conſider the manner of determining the values of annuities granted upon two or more lives; but before we enter upon this, it will be neceſſary (for the benefit of thoſe unacquainted with the principles of chances) to premiſe the following

LEMMA I.

If a ſum of money (S) *depends upon two different events, ſo as to be received, if both thoſe events happen; the probability of receiving that ſum, will be equal to the product of the probabilities of the happening of the two events.*

Let the number of chances for the happening of one of the events, be to the number of all the chances, both for its happening and failing, as *a* to *b*; and let the number of chances, for the happening of the other event, be to the number of all the chances, both for its happening and failing, as *c* to *d*; then it is manifeſt, that $\frac{c}{d} \times S$, or the given ſum diminiſhed in the ratio of *d* to *c*, would be the true value of the expectation upon this ſum (conſidered without regard to time) was the ſame to be received, or to depend only, upon the happening of the laſt named of the two propoſed events; or, in other words, $\frac{c}{d} \times S$, will be the ſum that might be taken as an equivalent in this caſe, for the chance of obtaining the ſum S. But the probability of the happening of the firſt event, or of being intitled to receive the ſum $\frac{c}{d} \times S$, as an equivalent being only $\frac{a}{b}$, the value of the expectation; therefore, as it depends on both events, can be only the $\frac{a}{b}$ part of this ſum, or $\frac{a}{b} \times \frac{c}{d} \times S$, and conſequently the probability of receiving the ſum S, only $\frac{a}{b} \times \frac{c}{d}$, *which was to be proved.*

COROLLARY I.

Hence, alſo, may the probability of receiving any ſum of money, depending on the happening of 3, 4, *&c.* events be eaſily derived: For let $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, $\frac{g}{h}$, *&c.* repreſent the probabilities of happening of ſo many different events; and S, T, U, W, *&c.* ſeveral ſums depending thereon, the firſt S, to be received upon the happening of the two firſt; the ſecond T, upon the happening of the three firſt; the third U, upon the happening of the four firſt, *&c.*

reckoning according to the order in which the reſpective probabilities are placed.

Then, ſeeing the probability of receiving the firſt ſum S, is $\frac{a}{b} \times \frac{c}{d}$, and that the expectation on the ſecond T, depends entirely upon the happening of this event, and that of the aforeſaid order, whoſe probability is $\frac{e}{f}$, it follows that $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f}$, the product of thoſe two, will be the probability of receiving the ſum T. In like manner, as the expectation on the ſum U, depends upon the receiving of T, and the happening of the event, whoſe probability is $\frac{g}{h}$, the probability of receiving the ſum U, will be $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} \times \frac{g}{h}$, &c. whence it appears, that the probability of the happening of any number of events, is equal to the product of all the probabilities of happening of thoſe events, conſidered ſeparately.

PROBLEM I.

The probability of life, and rate of intereſt being given; to find the value of an annuity, granted upon any number of joint *lives, that is, for as long as they ſhall all continue in being together.*

SOLUTION.

Let the propoſed annuity be 1 L, and A, B, C, *&c.* the lives upon which it is granted, and r the amount of 1 L, in one year, *viz.* principal and intereſt; let the probability of the life A continuing 1, 2, 3, *&c.* years, be repreſented by a, $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$, $\overset{\prime\prime\prime}{a}$, *&c.* reſpectively; and that of the life B continuing 1, 2, 3, *&c.* years, by b, $\overset{\prime}{b}$, $\overset{\prime\prime}{b}$, *&c.* *&c.* then will the probability of all the lives continuing; 1, 2, 3, *&c.* years, be $a\,b\,c\,d$, &c. $\overset{\prime}{a}\,\overset{\prime}{b}\,\overset{\prime}{c}\,\overset{\prime}{d}$, &c. $\overset{\prime\prime}{a}\,\overset{\prime\prime}{b}\,\overset{\prime\prime}{c}\,\overset{\prime\prime}{d}$, &c. &c. &c. reſpectively, by the Corollary to the preceeding Lemma. Theſe, therefore, being reſpectively multiply'd into the terms of the geometric progreſſion, $\frac{1}{r}$, $\frac{1}{r^2}$, $\frac{1}{r^3}$, *&c.* ſhewing the preſent value of 1 L certain, to be received at the expiration of 1, 2, 3, *&c.* years, the products $\frac{a\,b\,c\,d,\ \&c.}{r}$ $\frac{\overset{\prime}{a}\,\overset{\prime}{b}\,\overset{\prime}{c}\,\overset{\prime}{d},\ \&c.}{r^2}$ $\frac{\overset{\prime\prime}{a}\,\overset{\prime\prime}{b}\,\overset{\prime\prime}{c}\,\overset{\prime\prime}{d},\ \&c.}{r^3}$, &c. thence ariſing, will reſpectively expreſs the preſent values of the 1ſt, 2d, 3d, *&c.* year's rents, upon the contingency of one or more of the lives failing, in 1, 2, 3, *&c.* years: The ſum of all which, or $\frac{a\,b\,c\,d,\ \&c.}{r} + \frac{\overset{\prime}{a}\,\overset{\prime}{b}\,\overset{\prime}{c}\,\overset{\prime}{d},\ \&c.}{r^2} + \frac{\overset{\prime\prime}{a}\,\overset{\prime\prime}{b}\,\overset{\prime\prime}{c}\,\overset{\prime\prime}{d},\ \&c.}{r^3}$ &c. is, therefore, the preſent value of the annuity. Q. E. I.

COROLLARY I.

Hence, will the value of an annuity for one ſingle life A, be expreſs'd by $\frac{a}{r} + \frac{\overset{\prime}{a}}{r^2} + \frac{\overset{\prime\prime}{a}}{r^3} + \frac{\overset{\prime\prime\prime}{a}}{r^4}$, *&c.* for two joint lives, A B, by $\frac{ab}{r} + \frac{\overset{\prime}{a}\overset{\prime}{b}}{r^2} + \frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}}{r^3} + \frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}}{r^4}$, *&c.* and for three joint lives by $\frac{abc}{r} + \frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{r^2} + \frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{r^3} + \frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{r^4}$, *&c.*

COROLLARY II.

If a, $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$, $\overset{\prime\prime\prime}{a}$, &c. be taken equal to a, a^2, a^3, a^4, &c. and b, $\overset{\prime}{b}$, $\overset{\prime\prime}{b}$, $\overset{\prime\prime\prime}{b}$, &c. equal to b, b^2, b^3, b^4, &c. &c. then will the value of the annuity be defined by the geometric progreſſion $\frac{abcd, \&c.}{r} + \frac{a^2b^2c^2d^2, \&c.}{r^2} + \frac{a^3b^3c^3d^3, \&c.}{r^3}$, *&c.* or its equal $\frac{abcd, \&c.}{r - abcd, \&c.}$

COROLLARY III.

Therefore, if the value of an annuity for each of the ſingle lives be given, equal to M, N, P, Q, *&c.* reſpectively; then, according to the above hypotheſis, $\frac{a}{r-a}$, being $= M$, $\frac{b}{r-b} = N$, $\frac{c}{r-c} = P$, *&c.* we ſhall have $a = \frac{rM}{M+1}$, $b = \frac{rN}{N+1}$, $c = \frac{rP}{P+1}$, *&c.* and conſequently the value of an annuity for all the joint lives equal to $\frac{rM \times rN \times rP \times rQ, \&c.}{r \times \overline{M+1} \times \overline{N+1} \times \overline{P+1}, \&c. - rM \times rN \times rP, \&c.}$, upon a ſuppoſition that the probabilities of continuing in being 1, 2, 3, *&c.* years, are to one another, as the terms of a geometric progreſſion, or that the decrements of life are in a conſtant ratio.

COROLLARY IV.

But if the probability of living an aſſigned part of time, be ſuppoſed to decreaſe continually, to the extremity of old age, ſo that the terms of the ſeries a, $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$, $\overset{\prime\prime\prime}{a}$, &c. may be reſpectively expounded by thoſe of an arithmetic progreſſion, $\frac{m-1}{m}$, $\frac{m-2}{m}$, $\frac{m-3}{m}$, $\frac{m-4}{m}$, *&c.* and the terms of the ſeries b, $\overset{\prime}{b}$, $\overset{\prime\prime}{b}$, $\overset{\prime\prime\prime}{b}$, *&c.* by thoſe of an arithmetic progreſſion, $\frac{n-1}{n}$, $\frac{n-2}{n}$, $\frac{n-3}{n}$, $\frac{n-4}{n}$, *&c. &c.* then will the value of an annuity, upon all the joint lives, it is manifeſt, be

$\frac{m-1}{rm} \times \frac{n-1}{n} \times \frac{p-1,\ \&c.}{p} + \frac{m-2}{r^2m} \times \frac{n-2,\ \&c.}{n} + \frac{m-3}{r^3m} \times \frac{n-3,\ \&c.}{n}$, *&c.* where *m*, *n*, *p*, &c. reſpectively repreſent the numbers of years, which the ſeveral lives have a chance to continue, reckoning to the extremity of old age.

COROLLARY V.

Hence it appears that the value of an annuity, according to this laſt hypotheſis, for one ſingle life A will be $= \frac{m-1}{rm} + \frac{m-2}{r^2m} + \frac{m-3}{r^3m} + \frac{m-4}{r^4m}$, *&c.* for two joint lives A B, equal to $\frac{m-1 \times n-1}{rmn} + \frac{m-2 \times n-2}{r^2\,mn} + \frac{m-3 \times n-3}{r^3\,mn}$, *&c.* and for three joint lives, A, B, C, equal to $\frac{m-1 \times n-1 \times p-1}{rmnp} + \frac{m-2 \times n-2 \times p-2}{r^2\,mnp} + \frac{m-3 \times n-3 \times p-3}{r^3\,mnp}$, *&c.* each ſeries being to be continued to a number of terms (*m*,) equal to the number of years included between the oldeſt life A, and the extremity of old age.

But theſe ſeries may be ſummed, and will be found equal to $\frac{1}{r-1} - \frac{r+r^{1-m}}{m \times \overline{r-1}|^2}$, $\frac{1}{r-1} - \frac{\overline{m+n-1} \times r + \overline{n-m-1} \times r^{1-m}}{mn \times \overline{r-1}|^2} + \frac{2r-2r^{1-m}}{mn \times \overline{r-1}|^3}$, and $\frac{1}{r-1} - \frac{\overline{mnp - \overline{m-1\times n-1\times p-1}\times r}}{mnp\times\overline{r-1}|^{2}}$

$\frac{+\overline{n-m-1}\times\overline{p-m-1}\times r^{1-m}}{mnp\times\overline{r-1}|^{2}} + \frac{\overline{2m+2n+2p-}}{mnp\times\overline{r-1}|^{3}}$

$\frac{\overline{6}\times r + \overline{4m-2n-2p+6}\times r^{1-m}}{mnp\times\overline{r-1}|^{3}} +$

$\frac{6r^{1-m}-6r}{mnp\times\overline{r-1}|^{4}}$, reſpectively; which values, if M be put for the value $\left(\frac{1-r^{-m}}{r-1}\right)$ of an annuity, *certain*, for m years, and v for $\left(\frac{1}{r-1}\right)$ that of the ſame annuity for ever, will become $v - \frac{\overline{v+1}\times M}{m}$,

$v - \frac{v+1}{n}\times\overline{\overline{n-m-2v-1}\times\frac{M}{m}+2v}$,

and $v - \frac{v+1\times v}{np}\times\overline{2n+2p-m-6v-3}$

$+\frac{v+1}{np}\times\overline{n-m-1}\times\overline{p-m-1}\times\frac{M}{m}$

$-\frac{v+1\times 2v}{pn}\times\overline{n+p-2m-3v-3}\times\frac{M}{m}$,

reſpectively; ſhewing the worth of an annuity for one, two, or three joint lives; upon ſuppoſition, that the probabilities of living 1, 2, 3, *&c.* years, are to one another, as the terms of an arithmetic progreſſion; or that the decrements of life, from year to year, are all equal one to another.

COROLLARY VI.

But, let the probability of life be what it will, the required value may be always determined by help of a table of obſervations, and the general expreſſion foregoing; for let the number of the living, correſponding to the age of A, in the table, be repreſented by Q, and thoſe anſwering to the next ſucceeding ages, in the table, by $\overset{\prime}{Q}$, $\overset{\prime\prime}{Q}$, *&c.* reſpectively; and, in like manner, let the number of the living, anſwering to the age of B, be repreſented by R, and thoſe anſwering to the next ſucceeding ages, by $\overset{\prime}{R}$, $\overset{\prime\prime}{R}$, $\overset{\prime\prime\prime}{R}$, *&c.* *&c.* *&c.* then the probability, that the life A continues 1, 2, 3, *&c.* years, being $\frac{\overset{\prime}{Q}}{Q}$, $\frac{\overset{\prime\prime}{Q}}{Q}$, $\frac{\overset{\prime\prime\prime}{Q}}{Q}$, *&c.* and that of the life B, continuing 1, 2, 3, *&c.* years, equal to $\frac{\overset{\prime}{R}}{R}$, $\frac{\overset{\prime\prime}{R}}{R}$, $\frac{\overset{\prime\prime\prime}{R}}{R}$, *&c.* *&c.* we ſhall, by ſubſtituting theſe ſeveral values, inſtead of a, $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$, &c. b, $\overset{\prime}{b}$, $\overset{\prime\prime}{b}$, &c. in the general expreſſion, have $\frac{\overset{\prime}{Q}\overset{\prime}{R}\overset{\prime}{S},\ \&c.}{r\,QRS,\ \&c.} + \frac{\overset{\prime\prime}{Q}\overset{\prime\prime}{R}\overset{\prime}{S},\ \&c.}{r^2\,QRS,\ \&c.} + \frac{\overset{\prime\prime\prime}{Q}\overset{\prime\prime\prime}{R}\overset{\prime\prime\prime}{S},\ \&c.}{r^3\,QRS,\ \&c.}$, *&c.* or $\frac{1}{QRS,\ \&c.} \times \frac{\overset{\prime}{Q}\overset{\prime}{R}\overset{\prime}{S},\ \&c.}{r} + \frac{\overset{\prime\prime}{Q}\overset{\prime\prime}{R}\overset{\prime\prime}{S},\ \&c.}{r^2} + \frac{\overset{\prime\prime\prime}{Q}\overset{\prime\prime\prime}{R}\overset{\prime\prime\prime}{S},\ \&c.}{r^3}$ *&c.* equal to the value of the annuity.

COROLLARY VII.

Hence, if the value (P) of the joint lives A, B, C, be given, or once computed, the value (K) of the next younger lives, $\overset{\prime}{A}$, $\overset{\prime}{B}$, $\overset{\prime}{C}$, *&c.* whoſe ages are, each, reſpectively, one year leſs than thoſe of A, B, C, *&c.* may be eaſily derived; for let $\underset{\prime}{Q}$, $\underset{\prime}{R}$, $\underset{\prime}{S}$, *&c.* be the numbers found in the table of obſervations, againſt thoſe next younger ages; then, for the very ſame reaſons that $\frac{\overset{\prime}{Q}\overset{\prime}{R}\overset{\prime}{S}, \&c.}{rQRS, \&c.} + \frac{\overset{\prime\prime}{Q}\overset{\prime\prime}{R}\overset{\prime\prime}{S}, \&c.}{r^2 QRS, \&c.}$, *&c.* is $=$ P, ſhall $\frac{QRS, \&c.}{r\underset{\prime}{Q}\underset{\prime}{R}\underset{\prime}{S}, \&c.} + \frac{\overset{\prime}{Q}\overset{\prime}{R}\overset{\prime}{S}, \&c.}{r^2\underset{\prime}{Q}\underset{\prime}{R}\underset{\prime}{S}, \&c.}$, *&c.* be $=$ K: Wherefore, multiplying the former equation by QRS, *&c.* and the latter by $r\underset{\prime}{Q}\underset{\prime}{R}\underset{\prime}{S}$, *&c.* and, taking one from the other, we have QRS, *&c.* $= rK\underset{\prime}{Q}\underset{\prime}{R}\underset{\prime}{S}$, *&c.* — PQRS, *&c.* and conſequently $K = \frac{\overline{1+P}\times QRS, \&c.}{r\underset{\prime}{Q}\underset{\prime}{R}\underset{\prime}{S}, \&c.} = \overline{1+P}\times\frac{QRS, \&c.}{r\underset{\prime}{Q}\underset{\prime}{R}\underset{\prime}{S}, \&c.}$

COROLLARY VIII.

Laſtly, from Coroll. IV, V, and the given value (P) of the joint lives A, B, C, *&c.* the value of an annuity upon an equal number of other joint lives A', B', C', *&c.* reſpectively, younger than the former, by any number of years, during which, the decrements of life may be eſteemed equal, may be readily determined. Let s be the propoſed number of years, or the common difference between the ages of A and A', B and B', C and C', *&c.* and w $\left(=\frac{1}{r^s}\right)$ the preſent value of 1 L, due at the end of s years; let the number anſwering to each of the ages A, B, *&c.* be taken from the table of obſervations, and divided by the preceding decrement, and let the quotients (m, n, &c.) be conſidered as the complements of thoſe ages, to the extremity of old age, and let (S) the value of an annuity, anſwering to thoſe complements, by Coroll. V. be accordingly found.

Moreover, having added s to each of the ſaid quotients, and taken the ſums ($m+s$ $n+s$, *&c.*) thence ariſing, as the complements of the ages, A', B', C', *&c.* to the extremity of old age, let (T) the value of an annuity anſwering to theſe complements, in the ſame manner, be alſo found. Then it will follow, from what has been laid down in the forementioned Corollaries, that $T+\overline{P-S}\times\frac{w\,m\,n,\ \&c.}{\overline{m+s}\times\overline{n+s},\ \&c.}$ will expreſs the value of an annuity upon the joint lives, $\overset{\prime}{A}$, $\overset{\prime}{B}$, $\overset{\prime}{C}$, *&c.*

These two laſt Corollaries will be found very uſeful, in computing tables for the valuation of annuities upon one ſingle life, or 2, 3, or more *joint* lives, *as deducible from real obſervations*; and I have inſiſted more largely on this propoſition, becauſe the moſt intricate queſtions in the ſubject, may be referred to it, and readily ſolved by help of tables ſo computed; as, in the ſucceeding propoſitions, will be made to appear.

PROBLEM II.

The ſame things being given as in the laſt Propoſition; to find the value of an annuity, granted upon any number of aſſigned lives, that is, to continue as long as the longeſt of them is in being.

SOLUTION.

Let every thing be ſuppoſed as in the preceding problem; then ſince the probability that the life A, B, or C, *&c.* fails the firſt year, is expreſs'd by $1-a$, $1-b$, or $1-c$, *&c.* reſpectively, the probability of all the lives, A, B, C, *&c.* failing the firſt year, will be expreſs'd by $\overline{1-a} \times \overline{1-b} \times \overline{1-c}$, &c. (per Lemma) therefore that of ſome one or more of them, ſurviving the firſt year, will be $1-\overline{1-a} \times \overline{1-b} \times \overline{1-c}$, &c. In like manner, it will appear, that the probability of ſome one or more of the lives, ſurviving the ſecond year, will be $1-\overline{1-\overset{\prime}{a}} \times \overline{1-\overset{\prime}{b}}$, &c. &c. Therefore, if theſe ſeveral probabilities be reſpectively multiply'd into the terms of the geometric ſeries $\frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3}$, &c. expreſſing the preſent value of 1 L certain, to be received at the end of 1, 2, 3, &c. years, the products thence ariſing, will reſpectively exhibit the preſent values of the 1ſt, 2d, 3d, &c. year's rents, upon the contingency of ſome one or more of the lives ſurviving the 1ſt, 2d, 3d, &c. years; the ſum of all which products, or $\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}+\frac{1}{r^5}$, &c. $-\frac{\overline{1-a}\times\overline{1-b},\text{\&c.}}{r}-\frac{1-\overset{\prime}{a}\times 1-\overset{\prime}{b},\text{\&c.}}{r^2}-\frac{1-\overset{\prime\prime}{a}\times 1-\overset{\prime\prime}{b},\text{\&c.}}{r^3}-\frac{1-\overset{\prime\prime\prime}{a}\times 1-\overset{\prime\prime\prime}{b},\text{\&c.}}{r^4}$, is therefore the value of the annuity. Q. E. I.

COROLLARY I.

Hence, if the lives be all equal, and their number be repreſented by s, then $a\,b\,c$, &c. becoming equal to each other, the value of the annuity will be expreſſible by $\frac{1}{r}+\frac{1}{r^{2}}+\frac{1}{r^{3}}$, *&c.*

$$-\frac{\overline{1-a}\,\big|^{s}}{r}-\frac{\overline{1-a^{\prime}}\,\big|^{s}}{r^{2}}-\frac{\overline{1-a^{\prime\prime}}\,\big|^{s}}{r^{3}}-\frac{\overline{1-a^{\prime\prime\prime}}\,\big|^{s}}{r^{4}}-\frac{\overline{1-a^{\prime\prime\prime\prime}}\,\big|^{s}}{r^{5}},\ \textit{\&c.}$$

COROLLARY II.

Since the value of the annuity, converted to ſimple terms, is

$$\frac{a}{r}+\frac{b}{r},\ \textit{\&c.}-\frac{a\,b}{r}-\frac{a\,c}{r},\ \textit{\&c.}+\frac{a\,b\,c}{r}+\frac{a\,b\,d}{r},\ \textit{\&c.}$$

$$\frac{\overset{\prime}{a}}{r^{2}}+\frac{\overset{\prime}{b}}{r^{2}},\ \textit{\&c.}-\frac{\overset{\prime}{a}\,\overset{\prime}{b}}{r^{2}}-\frac{\overset{\prime}{a}\,\overset{\prime}{c}}{r^{2}},\ \textit{\&c.}+\frac{\overset{\prime}{a}\,\overset{\prime}{b}\,\overset{\prime}{c}}{r^{2}}+\frac{\overset{\prime}{a}\,\overset{\prime}{b}\,\overset{\prime}{d}}{r^{2}},\ \textit{\&c.}$$

$$\frac{\overset{\prime\prime}{a}}{r^{3}}+\frac{\overset{\prime\prime}{b}}{r^{3}},\ \textit{\&c.}-\frac{\overset{\prime\prime}{a}\,\overset{\prime\prime}{b}}{r^{3}}-\frac{\overset{\prime\prime}{a}\,\overset{\prime\prime}{c}}{r^{3}},\ \textit{\&c.}+\frac{\overset{\prime\prime}{a}\,\overset{\prime\prime}{b}\,\overset{\prime\prime}{c}}{r^{3}}+\frac{\overset{\prime\prime}{a}\,\overset{\prime\prime}{b}\,\overset{\prime\prime}{d}}{r^{3}},\textit{\&c.}$$

&c. &c. *&c. &c.* *&c. &c.*

where, according to the laſt propoſition, the firſt collateral column expreſſes the value of an annuity for the ſingle life A; the ſecond, the like for the ſingle life B; the third and fourth, for the joint lives A B and A C, *&c. &c.* it follows, that the value of an annuity, to continue as long as any one of the lives A, B, C, D, *&c.* is in being, is equal to the ſum of the values of all the ſingle lives, *leſs* the values of all the *joint* lives, combined two and two, *more* the values of all the *joint* lives, combined three and three, *leſs* the values of all the *joint* lives, combined four and four, and ſo on. Therefore, when the values of the *joint* lives are given, the value of an annuity upon the longeſt life, will from hence, be likewiſe given.

COROLLARY III.

But when the lives are all equal, the values of every 2, or 3, *&c.* *joint* lives, will likewiſe be equal; therefore, if the value of each ſingle life be repreſented by H, that of each two *joint* lives by $\overset{\prime}{H}$, that of each three joint lives by $\overset{\prime\prime}{H}$, *&c.* the values of all the *ſingle* lives, being s in number, will be $= sH$, and the values of all the *joint* lives, combined two and two, $= s \times \frac{s-1}{2}\overset{\prime}{H}$, *&c.* Whence it is manifeſt, that the value of the longeſt life, in this caſe, will be $sH - \frac{s}{1} \times \frac{s-1}{2}\overset{\prime}{H} + \frac{s}{1} \times \frac{s-1}{2} \times \frac{s-2}{3}\overset{\prime\prime}{H} - \frac{s}{1} \times \frac{s-1}{2} \times \frac{s-2}{3} \times \frac{s-3}{4}\overset{\prime\prime\prime}{H} + \frac{s}{1} \times \frac{s-1}{2} \times \frac{s-2}{3} \times \frac{s-3}{4} \times \frac{s-4}{5}\overset{\prime\prime\prime\prime}{H}$, *&c.*

COROLLARY IV.

If the probabilities of continuing in being 1, 2, 3, *&c.* years, be expounded by the terms of an arithmetic progreſſion, or the decrements of life from year to year, be ſuppoſed equal (as in Coroll. IV. and V. of the laſt Propoſition) and if *m n p*, *&c.* be reſpectively put for the numbers of years between the ſeveral ages A, B, C, *&c.* and the extremity of old age, and M N P, *&c.* be taken to repreſent the preſent values of an annuity certain for thoſe numbers of years, and v that of the annuity for ever; then will the value of one life A be equal to $v - \overline{v+1} \times \frac{M}{m}$, of two lives A B, equal to $v - \frac{v+1}{n} \times \overline{M + N - 2v + \overline{2v+1} \times \frac{M}{m}}$ of three lives A, B, C, $= v - \frac{v+1}{np} \times$ $\overline{\overline{m+1}|^2 + \overline{2m+3v+3} \times 2v} \times \frac{M}{m} - \frac{v+1}{p} \times$ $\overline{\overline{n+2v+1} \times \frac{N}{n} + P} + \frac{\overline{v+1} \times v}{np} \times$ $\overline{2n+m+6v+3}$; *&c.* which values, therefore, when the lives A, B, *&c.* are all equal, will become $v - \overline{v+1} \times \frac{M}{m}$, $v -$ $\frac{v+1}{m} \times \overline{\overline{2m+2v+1} \times \frac{M}{m} - 2v}$, and $v -$ $\frac{v+1}{m^2} \times \overline{\overline{m+1} \times \overline{3m+6v} + 6v^2 + 1} \times \frac{M}{m}$ $\overline{\frac{3v \times \overline{v+1} \times \overline{m+2v+1}}{m^2}}$, reſpectively.

Note. When the propoſed lives are unequal, or of different ages, A is to be taken as the oldeſt of them, B as the next oldeſt, and ſo on.

PROBLEM III.

To find the value of an annuity granted upon any number, n, *of lives,* A, B, C, &c. *but ſo as to continue only as long as a given number,* m, *of them are in being.*

SOLUTION.

Let P be the value of all the *joint* lives, A, B, C, *&c.* that is, the value of an annuity, for as long as they ſhall all continue in being together; and let Q be the ſum of the values of all the *joint* lives that can ariſe, by combining A, B, C, *&c.* ſo as to leave out one life at each combination, and R, the ſum of the values of all the *joint* lives, that can ariſe by combining the ſame, ſo as to leave out two lives at each combination, *&c, &c.* Then will

$$P\times\overline{1-\frac{n}{1}+n\times\frac{n-1}{2}-n\times\frac{n-1}{2}\times\frac{n-2}{3}}, \text{ \&c.}$$

$$Q\times\overline{1-\frac{n-1}{1}+\frac{n-1}{1}\times\frac{n-2}{2}-\frac{n-1}{1}\times\frac{n-2}{2}\times\frac{n-3}{3}}$$

$$R\times\overline{1-\frac{n-2}{1}+\frac{n-2}{1}\times\frac{n-3}{2}-\frac{n-2}{1}\times\frac{n-3}{2}\times\frac{n-4}{3}}$$

$$S\times\overline{1-\frac{n-3}{1}+\frac{n-3}{1}\times\frac{n-4}{2}-\frac{n-3}{1}\times\frac{n-4}{2}\times\frac{n-5}{3}}$$

&c.

be the value of the annuity required; where the firſt ſeries is to be continued to as many terms, as there are units in $n+1-m$; the ſecond to as many terms, as the firſt all but one; the third to as many as the ſecond, all but one, and ſo on.

COROLLARY.

Hence it appears, that the value of an annuity, to ceaſe upon the failing of the firſt life, will be P; upon the failing of the ſecond life, $\overline{1-n}\times P+Q$; upon failing of the third life, $\overline{1-n+n\times\frac{n-1}{2}}\times P+\overline{2-nQ}+R$; and upon failing of the fourth life, $\overline{1-n+n\times\frac{n-1}{2}-n\times\frac{n-1}{2}\times\frac{n-2}{3}}\times P+\overline{2-n+\frac{n-1}{1}\times\frac{n-2}{2}}\times Q+\overline{3-n}\times\overline{R}+S$, *&c. &c.*

PROBLEM IV.

To find the value of the reverſion of the longeſt of any number of lives, A, B, C, *after the longeſt of any number of other lives,* P, Q, R.

SOLUTION.

From the value of all the lives, A, B, C, P, Q, R, ſubtract the value of the lives, P, Q, R, in poſſeſſion; the remainder will be the value of the reverſion.

The truth of this Solution is almoſt ſelf-evident; for the exceſs of the value of all the lives, above that of the lives in poſſeſſion, is equal to the ſum that ought to be paid, for the chance of enjoying the annuity after the deceaſe of theſe laſt lives; and ſo muſt be the true value of the reverſion.

PROBLEM V.

To find the value of the reverſion of any number of joint lives, A, B, C, *after any number of joint lives,* P, Q, R.

SOLUTION.

From the value of the joint lives in reverſion, ſubtract the value of all the joint lives, and there will remain the value of the reverſion.

DEMONSTRATION.

Let the right of the reverſion, or all the rents that may happen to ariſe from the annuity, during the joint continuance of A, B, C, after one of the lives P, Q, R, in poſſeſſion, is extinct, belong entirely to one perſon K, and his heirs; and let him be admitted into immediate poſſeſſion of the annuity, for the joint lives, A, B, C, upon condition that he or his heirs ſhall pay back the rent thereof, till ſuch time as it becomes his or their own proper right. This being premiſed, it will appear, that as long as all the lives, A, B, C, P, Q, R, are in being, and no longer, ought K, or his heirs, to pay back the rents of the annuity; for firſt he ought to pay, while A, B, C, P, Q, R, are all in being, becauſe all this time he receives the rent of an annuity, to which he has no right; but ſecondly, he ought not to pay after the deceaſe of any of the lives, P, Q, R, ſince whatever he may happen to receive afterwards, is his own juſt property; nor ought he to pay after the deceaſe of any of the lives, A, B, C, becauſe then he is wholly exempted from all further benefit ariſing from the annuity. Therefore, ſeeing the value of all the rents that K and his heirs may happen to pay, is the ſame as the value of all the *joint* lives, A, B, C, P, Q, R, and the value of all that they may receive, or the whole produce of the annuity, the ſame as that of the *joint* lives, A, B, C, the Solution is manifeſt.

Otherwiſe,

Let the probabilities of the life A, continuing 1, 2, 3, *&c.* years, be denoted by a, $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$, &c. and thoſe of the life continuing 1, 2, 3, *&c.* years, by b, $\overset{\prime}{b}$, $\overset{\prime\prime}{b}$, &c. reſpectively, *&c.* In like manner, let the probabilities of the life P, continuing 1, 2, 3, *&c.* years, be denoted by p, $\overset{\prime}{p}$, $\overset{\prime\prime}{p}$, &c. and thoſe of the life Q continuing 1, 2, 3, *&c.* years, by q, $\overset{\prime}{q}$, $\overset{\prime\prime}{q}$, &c. let the annuity be 1 L, and m be the amount of 1 L in one year, *viz.* principal and intereſt. Now, the expectation of A, B, C, upon the firſt year's rent, depends upon theſe two events; firſt, that they all continue in being till the end of that year, and ſecondly, that ſome one, at leaſt, of the other lives, P, Q, R, fails before that time: Wherefore, ſeeing the probability of the former is abc, and that of the latter $1-pqr$, the probability that both happen, or that A, B, C, ſhall receive the firſt year's rent, will be $abc \times \overline{1-pqr}$, or $abc-abcpqr$; this, therefore, multiplied into $\frac{1}{m}$, the preſent value of 1 L, due at the end of one year, gives $\frac{abc}{m} - \frac{abcpqr}{m}$, for the true value of the expectation of A, B, C, upon the firſt year's rent; and by the very ſame way of reaſoning, the expectation of A, B, C, on the 2d, 3d, &c. year's rents, will appear to be $\frac{a'b'c'}{m^2} - \frac{a'b'c'p'q'r'}{m^2}$, $\frac{a''b''c''}{m^3} - \frac{a''b''c''p''q''r''}{m^3}$, &c. reſpectively; the ſum of all which, or $\frac{abc}{m} + \frac{a'b'c'}{m^2} + \frac{a''b''c''}{m^3}$, &c. $- \frac{abcpqr}{m} - \frac{a'b'c'p'q'r'}{m^2} - \frac{a''b''c''p''q''r''}{m^3}$, &c. is, therefore the total value of the reverſion; and this, from Problem I, appears to be equal to the value of the joint lives, A, B, C, leſs the value of all the joint lives, A, B, C, P, Q, R, *as was to be proved.*

PRO-

PROBLEM VI.

To find the value of the reverſion of any number of joint *lives,* A, B, C, *after the longeſt of any number of other lives,* P, Q, R.

SOLUTION.

Let a, b, c, *&c.* be as in the laſt Problem: Then, becauſe the probability of all the lives, A, B, C, continuing till the end of the firſt year, is abc, and that of all the other lives, P, Q, R, failing before that time, $\overline{1-p} \times \overline{1-q} \times \overline{1-r}$, the probability that both theſe events happen, or that A, B, C, receive the firſt year's rent, will (by the preceding Lemma) be $abc \times \overline{1-p} \times \overline{1-q} \times \overline{1-r}$; this therefore multiplied by $\frac{1}{m}$, the preſent value of 1 L certain, to be received at the end of one year, gives $\frac{abc \times \overline{1-p} \times \overline{1-q} \times \overline{1-r}}{m}$ for the true value of the expectation of A, B, C, upon the firſt year's rent, allowing for diſcount, and all contingencies. After the very ſame manner it will appear, that the value of the expectation of A, B, C, on the 2d, 3d, *&c.*

year's rents, will be $\frac{a'b'c'\times\overline{1-p'}\times\overline{1-q'}\times\overline{1-r'}}{m^2}$, and $\frac{a''b''c''\times\overline{1-p''}\times\overline{1-q''}\times\overline{1-r''}}{m^3}$, &c. reſpectively. Therefore the ſum of theſe values, or $\frac{abc\times\overline{1-p}\times\overline{1-q}\times\overline{1-r}}{m}+\frac{a'b'c'\times\overline{1-p'}\times\overline{1-q'}\times\overline{1-r'}}{m^2}$ $+\frac{a''b''c''\times\overline{1-p''}\times\overline{1-q''}\times\overline{1-r''}}{m^3}$, &c. is the whole value of the reverſion. Q. E. I.

COROLLARY.

If the laſt general expreſſion be reduced to ſimple terms, and compared with that in Problem I. it will appear that the value of the reverſion, of any number of *joint* lives A, B, C, after the longeſt of any number of lives, P, Q, R, is equal to the value of all the *joint* lives A, B, C, *leſs* the values of all the *joint* lives ariſing from combining (at each combination) all the lives A, B, C, with each *one* of the other; *more* the values of all the *joint* lives, ariſing from combining all the lives A, B, C, with each *two* of the other; *leſs* the values of all the *joint* lives, ariſing from combining all the lives A, B, C, with each *three* of the other, and ſo on.

PRO-

PROBLEM VII.

To find the value of the reverſion of the longeſt of any number of lives A, B, C, *after any number of* joint *lives*, P, Q, R.

SOLUTION.

Let m, a, a', b, b', p, p', q, &c. be ſtill as in the preceding propoſitions. Therefore, ſeeing the probability that one or more of the lives P, Q, R, fails the firſt year, is expreſſed by $1-pqr$, and that of one of the lives A, B, C, at leaſt, ſurviving the firſt year, by $1-\overline{1-a}\times\overline{1-b}\times\overline{1-c}$, the probability that theſe laſt A, B, C, receive the firſt year's rent, will be $\overline{1-pqr}\times\overline{1-\overline{1-a}\times\overline{1-b}\times\overline{1-c}}$, and conſequently the value of their expectation on that year's rent, $\overline{1-pqr}\times\dfrac{\overline{1-\overline{1-a}\times\overline{1-b}\times\overline{1-c}}}{m}$, or $\dfrac{1-\overline{1-a}\times\overline{1-b}\times\overline{1-c}}{m}-\dfrac{pqr\times\overline{1-\overline{1-a}\times\overline{1-b}\times\overline{1-c}}}{m}$. And, from the very ſame way of reaſoning, the value of their expectation on the 2d, 3d, &c. year's rents, will appear to be $\dfrac{1-\overline{1-a'}\times\overline{1-b'}\times\overline{1-c'}}{m^2}-\dfrac{p'q'r'\times\overline{1-\overline{1-a'}\times\overline{1-b'}\times\overline{1-c'}}}{m^2}$, and

$$\frac{1-\overline{1-a''}\times\overline{1-b''}\times\overline{1-c''}}{m^3}-\frac{p''q''r''\times\overline{1-\overline{1-a''}\times\overline{1-b''}\times\overline{1-c''}}}{m^3},$$

&c. Therefore the ſum all theſe, or

$$\frac{1-\overline{1-a}\times\overline{1-b}\times\overline{1-c}}{m}+\frac{1-\overline{1-a'}\times\overline{1-b'}\times\overline{1-c'}}{m^2}+$$

$$\frac{1-\overline{1-a''}\times\overline{1-b''}\times\overline{1-c''}}{m^3}, \text{\&c.}-\frac{pqr\times\overline{1-\overline{1-a}\times\overline{1-b}\times\overline{1-c}}}{m}$$

$$-\frac{p'q'r'\times\overline{1-\overline{1-a'}\times\overline{1-b'}\times\overline{1-c'}}}{m^2}-$$

$$\frac{p''q''r''\times\overline{1-\overline{1-a''}\times\overline{1-b''}\times\overline{1-c''}}}{m^3},$$ &c. muſt be the total value of the reverſion. Q. E. I.

COROLLARY.

Hence, becauſe it appears from the preceding problems, that the firſt ſeries, $\frac{1-\overline{1-a}\times\overline{1-b}\times\overline{1-c}}{m}+\frac{1-\overline{1-a'}\times\overline{1-b'}\times\overline{1-c'}}{m^2}$, &c. expreſſes the value of the longeſt of the lives A, B, C, and that the ſecond $\frac{pqr\times\overline{1-\overline{1-a}\times\overline{1-b}\times\overline{1-c}}}{m}$, &c. (reduced to ſimple terms) is equal to the values of all the joint lives ariſing from combining, at each combination, all the lives P, Q, R, with each *one* of the other, &c. it follows, that the value of the reverſion of any number of lives A, B, C, after any number of *joint* lives, P, Q, R, is equal to the value of the lives A, B, C; *leſs* the values of all the *joint* lives, ariſing from the combination of all the lives, in poſſeſſion with each *one* of the other; *more* the values of all the *joint* lives, ariſing from the combination of all the lives, in poſſeſſion with each *two* of the other; *leſs* the values of all the joint lives, ariſing from the combination of all the lives, in poſſeſſion with each *three* of the other, and ſo on.

TABLES

For the VALUATION of

ANNUITIES,

Upon One, Two, or Three
LIVES;

Deduced from ten Years OBSERVATIONS
on the BILLS of MORTALITY of the
City of *LONDON*.

TABLE I.

For the Valuation of Annuities upon one LIFE.

Age.	Year's Purch, at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*	Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*
6	14.1	16.2	18.8	21	12.9	14.7	17.0
7	14.2	16.3	18.9	22	12.7	14.5	16.8
8	14.3	16.4	19.0	23	12.6	14.3	16.5
9	14.3	16.4	19.0	24	12.4	14.1	16.3
10	14.3	16.4	19.0	25	12.3	14.0	16.1
11	14.3	16.4	19.0	26	12.1	13.8	15.9
12	14.2	16.3	18.9	27	12.0	13.6	15.6
13	14.1	16.2	18.7	28	11.8	13.4	15.4
14	14.0	16.0	18.5	29	11.7	13.2	15.2
15	13.9	15.8	18.3	30	11.6	13.1	15.0
16	13.7	15.6	18.1	31	11.4	12.9	14.8
17	13.5	15.4	17.9	32	11.3	12.7	14.6
18	13.4	15.2	17.6	33	11.2	12.6	14.4
19	13.2	15.0	17.4	34	11.0	12.4	14.2
20	13.0	14.8	17.2	35	10.9	12.3	14.1

Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*	Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*
36	10.8	12.1	13.9	56	8.4	9.1	10.1
37	10.6	11.9	13.7	57	8.2	8.9	9.9
38	10.5	11.8	13.5	58	8.1	8.7	9.6
39	10.4	11.6	13.3	59	8.0	8.6	9.4
40	10.3	11.5	13.2	60	7.9	8.4	9.2
41	10.2	11.4	13.0	61	7.7	8.2	8.9
42	10.1	11.2	12.8	62	7.6	8.1	8.7
43	10.0	11.1	12.6	63	7.4	7.9	8.5
44	9.9	11.0	12.5	64	7.3	7.7	8.3
45	9.8	10.8	12.3	65	7.1	7.5	8.0
46	9.7	10.7	12.1	66	6.9	7.3	7.8
47	9.5	10.5	11.9	67	6.7	7.1	7.6
48	9.4	10.4	11.8	68	6.6	6.9	7.4
49	9.3	10.2	11.6	69	6.4	6.7	7.1
50	9.2	10.1	11.4	70	6.2	6.5	6.9
51	9.0	9.9	11.2	71	6.0	6.3	6.7
52	8.9	9.8	11.0	72	5.8	6.1	6.5
53	8.8	9.6	10.7	73	5.6	5.9	6.2
54	8.6	9.4	10.5	74	5.4	5.6	5.9
55	8.5	9.3	10.3	75	5.2	5.4	5.6

TABLE II.

For the Valuation of Annuities upon two joint LIVES.

Mean Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*	Mean Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*
6	11.3	12.7	14.4	21	10.0	11.2	12.6
7	11.5	12.9	14.6	22	9.8	11.0	12.4
8	11.6	13.0	14.7	23	9.7	10.8	12.2
9	11.6	13.0	14.7	24	9.5	10.6	12.0
10	11.6	13.0	14.7	25	9.4	10.5	11.8
11	11.5	12.9	14.6	26	9.2	10.3	11.6
12	11.4	12.8	14.5	27	9.1	10.1	11.4
13	11.3	12.7	14.3	28	8.9	9.9	11.2
14	11.2	12.5	14.1	29	8.8	9.8	11.0
15	11.0	12.3	13.9	30	8.6	9.6	10.8
16	10.8	12.1	13.7	31	8.5	9.4	10.6
17	10.7	11.9	13.5	32	8.3	9.2	10.4
18	10.5	11.7	13.2	33	8.2	9.1	10.2
19	10.3	11.5	13.0	34	8.1	8.9	10.0
20	10.1	11.3	12.8	35	8.0	8.8	9.9

Mean Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*	Mean Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*
36	7.8	8.6	9.7	56	5.6	6.1	6.7
37	7.6	8.4	9.5	57	5.5	6.0	6.6
38	7.5	8.3	9.3	58	5.4	5.8	6.4
39	7.4	8.2	9.2	59	5.3	5.7	6.3
40	7.3	8.1	9.1	60	5.2	5.6	6.1
41	7.2	8.0	8.9	61	5.1	5.5	6.0
42	7.1	7.8	8.7	62	5.0	5.4	5.9
43	7.0	7.7	8.6	63	4.9	5.3	5.7
44	6.9	7.6	8.5	64	4.8	5.1	5.5
45	6.7	7.4	8.3	65	4.7	5.0	5.4
46	6.6	7.3	8.2	66	4.6	4.9	5.3
47	6.5	7.2	8.1	67	4.5	4.8	5.1
48	6.4	7.1	7.9	68	4.4	4.6	4.9
49	6.3	7.0	7.8	69	4.3	4.5	4.8
50	6.2	6.8	7.6	70	4.2	4.4	4.6
51	6.1	6.7	7.4	71	4.1	4.3	4.5
52	6.0	6.6	7.3	72	3.9	4.1	4.3
53	5.9	6.5	7.2	73	3.8	4.0	4.2
54	5.8	6.3	7.0	74	3.7	3.8	4.0
55	5.7	6.2	6.9	75	3.6	3.7	3.8

TABLE III.

For the Valuation of Annuities upon the longeſt of two LIVES.

Mean Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*	Mean Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*
6	16.9	19.7	23.3	21	15.6	18.2	21.3
7	17.0	19.8	23.4	22	15.4	18.0	21.1
8	17.1	19.9	23.5	23	15.3	17.8	20.8
9	17.1	19.9	23.5	24	15.1	17.6	20.6
10	17.1	19.9	23.5	25	15.0	17.4	20.3
11	17.1	19.9	23.5	26	14.9	17.3	20.1
12	17.0	19.8	23.4	27	14.7	17.1	19.9
13	16.9	19.7	23.3	28	14.6	16.9	19.7
14	16.7	19.5	23.1	29	14.5	16.8	19.5
15	16.6	19.3	22.9	30	14.4	16.6	19.3
16	16.4	19.1	22.6	31	14.2	16.4	19.1
17	16.2	18.9	22.4	32	14.1	16.2	18.9
18	16.1	18.7	22.1	33	14.0	16.1	18.7
19	15.9	18.5	21.9	34	13.9	15.9	18.5
20	15.7	18.3	21.6	35	13.8	15.8	18.3

Mean Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*	Mean Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*
36	13.7	15.6	18.1	56	11.2	12.1	13.4
37	13.6	15.5	17.9	57	11.0	11.9	13.1
38	13.5	15.3	17.7	58	10.9	11.7	12.8
39	13.4	15.2	17.5	59	10.7	11.5	12.5
40	13.3	15.0	17.3	60	10.5	11.2	12.2
41	13.2	14.9	17.0	61	10.3	11.0	12.0
42	13.1	14.7	16.8	62	10.1	10.8	11.7
43	13.0	14.5	16.5	63	9.9	10.5	11.4
44	12.9	14.3	16.3	64	9.7	10.3	11.1
45	12.8	14.2	16.1	65	9.4	10.0	10.8
46	12.6	14.0	15.8	66	9.2	9.7	10.5
47	12.5	13.8	15.6	67	8.9	9.4	10.2
48	12.4	13.6	15.3	68	8.7	9.2	9.9
49	12.2	13.4	15.1	69	8.5	8.9	9.5
50	12.1	13.3	14.9	70	8.2	8.6	9.2
51	11.9	13.1	14.6	71	8.0	8.4	8.9
52	11.8	12.9	14.4	72	7.7	8.1	8.6
53	11.6	12.7	14.1	73	7.5	7.8	8.2
54	11.5	12.5	13.9	74	7.2	7.5	7.9
55	11.3	12.3	13.6	75	6.9	7.2	7.6

TABLE IV.

For the Valuation of Annuities upon three joint LIVES.

Mean Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*	Mean Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*
6	9.7	10.6	11.7	21	8.2	9.0	10.0
7	9.9	10.8	11.9	22	8.1	8.9	9.8
8	10.0	10.9	12.0	23	7.9	8.7	9.6
9	10.0	10.9	12.0	24	7.7	8.5	9.4
10	10.0	10.9	12.0	25	7.6	8.3	9.2
11	9.9	10.8	11.9	26	7.4	8.1	9.0
12	9.8	10.7	11.8	27	7.3	8.0	8.8
13	9.6	10.5	11.6	28	7.1	7.8	8.6
14	9.5	10.4	11.4	29	7.0	7.7	8.5
15	9.3	10.2	11.2	30	6.8	7.5	8.3
16	9.2	10.0	11.0	31	6.7	7.4	8.2
17	9.0	9.8	10.8	32	6.5	7.2	8.0
18	8.8	9.6	10.6	33	6.4	7.1	7.9
19	8.6	9.4	10.4	34	6.2	6.9	7.7
20	8.4	9.2	10.2	35	6.1	6.8	7.6

Mean Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*	Mean Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*
36	6.0	6.7	7.4	56	4.4	4.7	5.1
37	5.9	6.5	7.2	57	4.3	4.6	5.0
38	5.8	6.4	7.1	58	4.2	4.5	4.9
39	5.7	6.3	7.0	59	4.1	4.4	4.8
40	5.6	6.2	6.9	60	4.0	4.3	4.6
41	5.5	6.1	6.8	61	3.9	4.2	4.5
42	5.4	6.0	6.7	62	3.8	4.1	4.4
43	5.4	5.9	6.5	63	3.7	4.0	4.3
44	5.3	5.8	6.4	64	3.7	3.9	4.2
45	5.2	5.7	6.3	65	3.6	3.8	4.1
46	5.1	5.6	6.2	66	3.5	3.7	3.9
47	5.0	5.5	6.1	67	3.4	3.6	3.8
48	5.0	5.4	5.9	68	3.3	3.5	3.7
49	4.9	5.3	5.8	69	3.2	3.4	3.6
50	4.8	5.2	5.7	70	3.1	3.2	3.4
51	4.7	5.1	5.6	71	3.0	3.1	3.3
52	4.7	5.1	5.5	72	2.9	3.0	3.1
53	4.6	5.0	5.4	73	2.8	2.9	3.0
54	4.5	4.9	5.3	74	2.6	2.7	2.8
55	4.4	4.8	5.2	75	2.5	2.6	2.7

TABLE V.

For the Valuation of Annuities upon the longeſt of three LIVES.

Mean Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*	Mean Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*
6	18.0	21.0	25.0	21	16.9	19.5	23.1
7	18.1	21.1	25.1	22	16.8	19.4	22.8
8	18.2	21.2	25.2	23	16.6	19.2	22.6
9	18.2	21.2	25.2	24	16.5	19.0	22.3
10	18.2	21.2	25.2	25	16.4	18.8	22.1
11	18.2	21.2	25.2	26	16.3	18.7	21.9
12	18.1	21.1	25.1	27	16.1	18.5	21.6
13	18.0	21.0	25.0	28	16.0	18.3	21.4
14	17.9	20.9	24.8	29	15.9	18.2	21.2
15	17.8	20.7	24.6	30	15.8	18.0	21.0
16	17.6	20.5	24.3	31	15.6	17.8	20.8
17	17.5	20.3	24.1	32	15.5	17.7	20.6
18	17.3	20.1	23.8	33	15.4	17.6	20.4
19	17.2	19.9	23.5	34	15.3	17.4	20.2
20	17.0	19.7	23.3	35	15.2	17.3	20.0

Mean Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*	Mean Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*
36	15.1	17.2	19.9	56	12.6	13.7	15.1
37	15.0	17.0	19.7	57	12.5	13.5	14.8
38	14.9	16.9	19.5	58	12.3	13.2	14.5
39	14.8	16.7	19.3	59	12.1	12.9	14.1
40	14.7	16.6	19.1	60	11.9	12.7	13.8
41	14.6	16.4	18.9	61	11.7	12.5	13.5
42	14.5	16.3	18.7	62	11.5	12.2	13.1
43	14.4	16.2	18.5	63	11.3	11.9	12.8
44	14.3	16.0	18.2	64	11.0	11.6	12.5
45	14.2	15.9	18.0	65	10.8	11.4	12.2
46	14.1	15.7	17.7	66	10.5	11.1	11.8
47	13.9	15.5	17.5	67	10.2	10.8	11.5
48	13.8	15.3	17.2	68	9.9	10.5	11.2
49	13.7	15.1	17.0	69	9.6	10.2	10.9
50	13.5	14.9	16.7	70	9.3	9.9	10.5
51	13.4	14.7	16.5	71	9.0	9.6	10.2
52	13.2	14.5	16.2	72	8.7	9.2	9.8
53	13.1	14.3	15.9	73	8.4	8.9	9.5
54	12.9	14.1	15.7	74	8.1	8.6	9.1
55	12.8	13.9	15.4	75	7.8	8.2	8.7

Here follow the practical Solutions of ſeveral Problems, depending on the foregoing tables.

PROBLEM VIII.

To find the value of an annuity for an aſſigned life.

SOLUTION.

Look out the given age in table I. and againſt it, towards the right-hand, under the propoſed rate of intereſt, will ſtand the number of years purchaſe, which an annuity upon that life is worth.

EXAMPLE.

Let the given age be 18 years, and the rate of intereſt 4 *per cent.* then looking againſt 18, under 4 *per cent.* I find 15.2, equal the number of years purchaſe required.

PROBLEM IX.

To find the value of an annuity upon two aſſigned joint *lives.*

SOLUTION.

CASE I.

If the two lives be equal; enter tab. II. with the common age, and againſt it you will have the value required.

CASE II.

If the given ages be unequal, but neither of them leſs than 25, nor greater than 50 years; take half the ſum of the two for a mean age, and proceed as in Caſe I *.

CASE III.

If one or both ages be without the limits abovementioned, but ſo that the difference of the values correſponding to thoſe ages, be not more than $\frac{1}{3}$ of the leſſer; let $\frac{4}{10}$ of that difference be added to the ſaid leſſer value, and the ſum will be the value ſought.

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* This and the following Solutions are ſo contrived, as to be always depended on to leſs than $\frac{1}{4}$ of a year's purchaſe, as I ſhall hereafter endeavour to make appear.

Generally,

Be the difference of the values what it will, multiply it by $\frac{1}{2}$ the leſſer of the two values, dividing the product by the greater; then the quotient, added to the leſſer value, will give the true anſwer very near.

EXAMPLE *of* CASE I.

Let the two given ages be each 18, and intereſt at 5 *per cent.* then in tab. II. againſt 18, under 5 *per cent.* is 10.5 years purchaſe.

EXAMPLE *of* CASE II.

In which the rate of intereſt is ſuppoſed as above, and one of the two ages 34, the other 48; therefore the half ſum of the ages is 42, againſt which ſtands 7.1.

EXAMPLE *of* CASE III.

Where one age is ſuppoſed to be 15 years, the other 29; here againſt 15 years will be found 11.0, and againſt 29, 8.8, the difference of which two values is 2.2, and $\frac{4}{10}$ thereof, equal to 0.88; this therefore, added to 8.8, gives 9.68, or 9.7, for the anſwer.

EXAMPLE *of* CASE IV.

Let the rate of intereſt be 4 *per cent.* and one age 11 years, the other 68. The values correſponding to theſe ages, are 12.9. and 4.6, their difference is 8.3, which multiplied by 2.3. will be 19.09, this divided by 12.9, quotes 1.5, which therefore, added to 4.6, the leſſer value, gives 6.1, equal the value ſought.

PROBLEM X.

To find the value of an annuity upon two lives, that is, to continue as long as either of them is in being.

SOLUTION.

CASE I.

If the lives be equal, find the given age in tab. III. and againſt it, under the propoſed rate of intereſt, will be the number of years purchaſe required.

CASE II.

If both ages be between 25 and 50, take half their ſum for a *mean age*, and proceed as in Caſe I.

CASE III.

If one or both ages be without the limits mentioned in the laſt caſe, but the difference of values correſponding to thoſe ages, as found in tab. III. be not more than $\frac{1}{6}$ part of the leſſer; take half the ſum of thoſe values for the value required.

Generally,

Let the given ages be what they will, find the value of the two joint lives by Caſe IV. Prob. IX. which ſubtract from the ſum of the values of the two ſingle lives, and there will remain the required value of an annuity upon the longeſt life.

EXAMPLE *of* CASE I.

Wherein the two given ages are each ſuppoſed 50 years, and the rate of intereſt 4 *per cent*. Here againſt 50 years, in tab. III. under 4 *per cent.* ſtands 13.3, ſhewing the number of years purchaſe, which an annuity is worth for two ſuch lives.

EXAMPLE *of* CASE II.

Suppoſe one age 30 years, and the other 46; then, the half ſum of the ages will be 38, anſwering to which, under 4 *per cent.* ſtands 15.3.

EXAMPLE *of* CASE III.

Let the two propoſed ages be 6, and 21 years; then againſt 6 years will be 19.7, and againſt 21, 18.2, the half ſum whereof, is 18.95, equal to the number of years purchaſe required,

EXAMPLE *of* CASE IV.

Let one age be 11 years, the other 68, and the rate of intereſt as in the preceding examples. Then the value of the two *joint* lives, by Caſe IV. of the laſt Problem, will be found 6.1, and the values of the ſingle lives, by Problem VIII. equal to 16.3, and 6.7, the ſum of which two, decreaſed by 6.1, is 16.9, equal to the value required.

PROBLEM XI.

To find the value of an annuity upon three joint *lives.*

SOLUTION.

CASE I.

If all the lives be equal; find out the given age in tab. IV. and againſt it, under the propoſed rate of intereſt, will be the number of years purchaſe required.

CASE II.

If all the three ages be between 15 and 55 years, and the difference between the greateſt and leaſt of them not more than 15 years, take $\frac{1}{3}$ part of their ſum for the *mean age*, and proceed as in Caſe I.

CASE III.

If one or more of the propoſed ages be without the limits, mentioned in the laſt article, but the difference of the values anſwering to the greateſt and leaſt of them, not greater than half the leaſt; then to the ſum of the two greater values, add twice the leaſt, and take $\frac{1}{4}$ of the ſum for a mean value required.

Generally,

Be the ages what they will, multiply the ſum of the three correſponding values, by the ſquare of the leaſt of them, reſerving the product; multiply the two greater values into each other, and to the double of the product, add the ſquare of the leſſer value; divide the reſerved product by this ſum, and ſubtract the quotient from twice the leſſer value; the reſult will be the value ſought.

EXAMPLE *of* CASE I.

Let each age be 35, and the rate of intereſt 3 *per cent*. then in tab. IV, againſt 35, under 3 *per cent*. ſtands 7.6, which is the number of years purchaſe that an annuity is worth for the three *joint* lives.

EXAMPLE *of* CASE II.

Let the three given ages be 20, 25, and 33 years. Here $\frac{1}{3}$ of the ages will be 26, correſponding to which, under 3 *per cent*, ſtands 9.0.

EXAMPLE *of* CASE III.

Where the propoſed ages are 7, 15, and 33 years; againſt theſe ſtand 11.9, 11.2, and 7.9, therefore the ſum of the two greater values is, here, 23.1; this added to twice the leſſer, gives 38.9, the $\frac{1}{4}$ of which, or 9.725, is the value ſought.

EXAMPLE *of* CASE IV.

Let the three ages be 13, 31 $\frac{1}{2}$, and 53 years, and intereſt 4 *per cent*. then the values anſwering to thoſe ages, will be 10.5, 7.3, and 5.0; the ſum whereof is 22.8, which multiply'd by 25, the ſquare of the leaſt of them gives 570, to be reſerved: Again, the two greateſt values multiply'd into each other produce, 76.65, the double of this added to 25, the ſquare of the leaſt will be 178.3, by which dividing 570, the reſerved product, there comes out 3.2; this ſubtracted from 10, the double of the leaſt value, leaves 6.8 for the value required.

PRO-

PROBLEM XII.

To find the value of an annuity upon the longeſt of three lives.

SOLUTION.

CASE I.

If the lives be all equal, ſeek the common age in tab. V. and againſt it, under the propoſed rate of intereſt, will be the number of years purchaſe required.

CASE II.

If none of the ages be leſs than 10, nor greater than 60 years, and the difference between the greateſt and leaſt of them not more than 15 years, to twice the ſum of the two leaſt add the greateſt, and take $\frac{1}{5}$ part of the ſum as a *mean age*.

CASE III.

If the difference of the greateſt and leaſt values, found againſt the propoſed ages in tab. V. be not more than $\frac{1}{4}$ of the leaſt; then, to twice the ſum of the two greateſt values, add the leaſt; taking $\frac{1}{5}$ part of the ſum for a *mean value*,

Generally,

Find the value anſwering to the greateſt of the given ages in tab. III. and the values correſponding to all the three ſeveral ages in tab. V. and let the difference of the two values, anſwering to the greateſt age, be taken and reſerved; let the ſquare of the greater of theſe two, be divided by the product of the two other remaining values; multiply the ſquare of the quotient by the reſerved difference, then this laſt product, added to the value of an annuity for the two youngeſt lives, will be the value required.

EXAMPLE *of* CASE I.

Let the three ages be each 35 years, and intereſt 4 *per cent*, then in tab. V. againſt 35, under 4 *per cent*. ſtands 17.3, for the number of years purchaſe required.

EXAMPLE *of* CASE II.

Let the propoſed ages be 16, 24, and 30 years, then will the *mean age* be 22 years, and the number of years purchaſe required 19.4.

Ex-

EXAMPLE *of* CASE III.

Suppoſe the three ages to be 28, 35, and 44, then the three correſponding values will be 18.3, 17.3, and 16.0, and therefore twice the ſum of the two greater added to the leſſer, is 87.2, which divided by 5, quotes 17,44 for the anſwer.

EXAMPLE *of* CASE IV.

Let the given ages be 20, 36, and 60, and intereſt as in the preceding examples: Here, the value found againſt 60 years in tab. III. is 11.2, and thoſe againſt 20, 36, and 60, in tab. V. 19.7, 17.2, and 12.7, reſpectively; wherefore, taking 11.2 from 12.7, we have 1.5 for the difference to be reſerved: Now the ſquare of 12.7, divided by the product of 19.7, and 17.2 is 0.5, the ſquare of which, multiply'd by 1.5, the reſerved difference gives 0.375; this added to 17.0, the value of an annuity for the two youngeſt lives (as determined by Caſe II. Prob. X.) will give 17.375, or 17.4 for the number of years purchaſe, which an annuity is worth upon all the three lives.

REMARK.

That the reader may not entertain any ſcruple concerning the exactneſs of the methods of Solution hitherto laid down, for eſtimating the values of annuities upon two or more unequal lives, I ſhall here, according to my promiſe, endeavour to make it appear, that thoſe Solutions may be always depended on, as very near the truth. In order to this, it will be requiſite to reſume the two hypotheſes laid down in Corol. II. and IV. Prob. I. wherein the probabilities of life are ſuppoſed in a geometrical, and in an arithmetical progreſſion; and to compare the values of *equal* fictitious lives, computed according to thoſe hypotheſes, with the correſponding values in the tables, for real lives, computed from actual obſervations, and then to conſider from thence, how the values ought to differ in lives that are *unequal*. Accordingly, let the value of each of the equal lives, whether conſidered as real or fictitious, be ſuppoſed equal to any number of years purchaſe, as 7, 8, 9, 10, 11, 12, 13, 14, and 15, ſucceſſively, and let the rate of intereſt be at 4 *per cent*. then will the correſponding value, of two equal joint lives, be as in the following little table, whereof the firſt column expreſſes the value of each of the ſingle lives, and the 2d, 3d, and 4th columns, the value of the joint lives, according to obſervations, and the two foreſaid hypotheſes reſpectively.

Value of one ſingle life.	Value of 2 joint lives, per tab.	Value of 2 joint lives, per 1ſt hypoth.	Value of 2 joint lives, per 2d hypoth.
7	4.7	3.9	4.9
8	5.4	4.6	5.6
9	6.1	5.3	6.4
10	6.8	6.1	7.1
11	7.6	6.9	7.9
12	8.5	7.8	8.8
13	9.5	8.7	9.7
14	10.5	9.7	10.6
15	11.5	10.6	11.6

Now, by inſpecting this table, we may obſerve, firſt, that the value of the joint lives, according to the laſt of the two hypotheſes, is a ſmall matter greater than the value of the ſame lives, as deduced from real obſervations, but never by more than about $\frac{3}{10}$ of an year's purchaſe; and ſecondly, that, on the other hand, the value of the joint lives, according to the firſt hypotheſis, is always leſs than the true value deduced from obſervations, and that at leaſt by $\frac{7}{10}$ of a year's purchaſe. Hence we may infer, that the probabilities of life, as given in the table of obſervations, do not come ſo near a geometric progreſſion, as to an arithmetic one (which, in ſome meaſure, appears from the table itſelf) and conſequently that the value of an annuity upon real lives, whether equal or unequal, will differ little from the value derived from the laſt hypotheſis, but ſomething more from the former. Let us, therefore, now ſee what the differences will be, in two unequal joint lives, by the general rule before given, (in Prob. IX.) from whence we ſhall be enabled to judge of the exactneſs of that rule. What theſe differences are, may be ſeen by the following table, which exhibits the values of the joint lives, according to each of the three foreſaid ways; wherein the value computed by the rule, compared with thoſe derived from the hypotheſes, appears to agree ſo exactly, throughout the whole table, with what has been above obſerved, with reſpect to the true value, as to ſufficiently prove, that the rule it ſelf muſt be very near the truth. But if this rule be near the truth, the two particular ones preceding it, muſt be ſo too, being ſo contrived, as to always bring out nearly the ſame value with the general one; but with this difference, that, as the general one, for the moſt part, gives the anſwer a little too ſmall, the firſt of theſe always makes it a little too great; tho' neither of them ſcarce ever err, by more than $\frac{1}{10}$ of a year's purchaſe.

Values of the two ſingle lives.	Value of the two joint lives, per rule.	Value of the two joint lives, per firſt hypoth.	Value of the two joint lives, per ſecond hypoth.
6 and 8	4.5	3.8	4.7
6 10	4.8	4.3	5.0
6 12	5.0	4.6	5.2
6 14	5.2	4.9	5.4
6 16	5.4	5.2	5.5
8 10	6.0	5.3	6.2
8 12	6.4	5.8	6.7
8 14	6.7	6.3	7.0
8 16	6.9	6.7	7.2
10 12	7.5	6.8	7.8
10 14	8.0	7.5	8.3
10 16	8.4	8.1	8.7
12 14	9.3	8.6	9.5
12 16	9.9	9.4	10.1
14 16	11.4	10.6	11.5

In the ſame manner it may be made to appear, that the other rules for three *joint* lives, and the *longeſt* of two or three lives, are likewiſe very near the truth; but I ſhall content myſelf here, with giving one or two inſtances, in annuities upon three joint lives. Let there be three equal lives, and the value of an annuity upon each of them 14 years purchaſe, and intereſt at 4 *per cent.* then will the value of the joint lives, by tab. IV. come out 8.3, but by the two hypotheſes, 7.3, and 8.5, reſpectively. Again, let the lives be ſuppoſed very unequal, ſo as to be worth 6, 10, and 16 years purchaſe, then will the value of the joint lives be, by the general rule, 4.5, but according to the hypotheſes, 3.84, and 4.63; which examples, agreeing ſo well with each other, and with what has been aboveſaid, tend greatly to evince the accuracy of the rules, or, at leaſt, to ſhew that they are conſiſtent with the table of obſervations.

OF

REVERSIONS.

PROBLEM XIII.

TO find the value of the Reverſion of one life after another.

SOLUTION.

From the value of the life in expectation, ſubtract the value of the two *joint* lives, or from the value of the *longeſt* of the two lives, ſubtract the value of the life in poſſeſſion; the remainder, in either caſe, will be the required value of the Reverſion *.

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* This and all the following Solutions, relating to Reverſions, are univerſally true, be the probability of life what it will, as appears from Prob. IV. V. and VI. they being nothing more than the moſt uſeful Caſes of the general Theorems there given.

EXAMPLE.

Suppoſe the age of the life in poſſeſſion to be 68 years, and that of the life in expectation 11 years, and intereſt 4 *per cent.* then the values of the two *joint* lives, by Caſe IV. Prob. IX. will be 6.1, which ſubtracted from 16.3, the value of the life in expectation, leaves 10.2, for the value of the reverſion; but if the youngeſt life had been in poſſeſſion, the value of the Reverſion would have been only 0.6.

PROBLEM XIV.

To find the value of the Reverſion of two lives after one.

SOLUTION.

From the value of the three lives, ſubtract the value of the life in poſſeſſion; the remainder will be the value of the two lives in Reverſion.

EXAMPLE.

Let the age of the life in poſſeſſion be 50 years, and thoſe of two lives in reverſion, 45 and 56 years, and intereſt at 4 *per cent.* then the value of the three lives, by Caſe II. Prob. XII. will be 15.1, from which ſubtracting 10.1, the value of the life in poſſeſſion, there remains 5.0, for the value required.

PROBLEM XV.

To find the value of the Reverſion of one life after two.

SOLUTION.

From the value of the three lives, take the value of the two lives in poſſeſſion, there will remain the value of the life in reverſion.

EXAMPLE.

Suppoſe 18 and 26 to be the ages of the two lives in poſſeſſion, and 32 that of the life in expectation, and intereſt at 4 *per cent.* then the value of the three lives, by Caſe II. Prob. XII. will be 19.0, from which ſubtracting 18.0, the value of the two lives in poſſeſſion, there remains 1.0, for the value of the reverſion.

PROBLEM XVI.

To find the value of the reverſion of one life after two joint *lives.*

SOLUTION.

From the value of the life in expectation, ſubtract the value of the three *joint* lives, there will remain the value of the life in reverſion.

EXAMPLE.

Let the age of each of the three propoſed lives, be 21 years; then the value of the three *joint* lives, by Caſe I. Prob. XI. will be 9.0, which ſubtracted from 14.7, the value of the life in expectation, leaves 5.7, equal the value of the reverſion, when intereſt is at 4 *per cent.*

PROBLEM XVII.

To find the value of the reverſion of two joint lives after one.

SOLUTION.

From the value of the two *joint* lives, ſubtract the value of the three joint lives; the remainder will be the value of the reverſion.

EXAMPLE.

Suppoſe the age of the life in poſſeſſion to be 16 years, and the ages of the two joint lives in reverſion, each 28 years; then the value of the two *joint* lives will be 9.9, by Caſe I. Prob. IX. and that of the three *joint* lives, 8.5, by Caſe II. Prob. XI. the difference of which, or 1.4, is the value of the reverſion; intereſt being 4 *per cent.*

PROBLEM XVIII.

Suppoſing two perſons, A, B, *to be equally in poſſeſſion of an annuity, which after the deceaſe of either of them, is to belong entirely to the ſurvivor for life: To find the value of the ſhare of each in that annuity.*

SOLUTION.

From the value of the life A, or B, ſubtract half the value of the two *joint* lives; the remainder will be the value of the ſhare of A, or B.

EXAMPLE.

Let the age of A be 18 years, and that of B, 29 years, and the rate of intereſt 4 *per cent.* then the value of the life A will be 15.2, and the value of the life B 13.2, from each of which ſubtracting (5.3) half the value of the two *joint* lives, there will remain 9.9, and 7.9, equal to the two values required.

PROBLEM XIX.

A *and* B *enjoy an annuity, to which a third perſon* C, *after the deceaſe of* A, *is to have the ſole right of poſſeſſion for life, provided* B *be then extinct; otherwiſe it is to be equally divided between him and* B, *during their joint lives, and then to belong entirely to* C, *for life, if he be the laſt ſurvivor: To find the value of the right of* C *in that annuity.*

SOLUTION.

From the value of the life C, ſubtract half the value of the two joint lives B, C, and from the value of the two joint lives A, C, ſubtract half the value of the three *joint* lives A, B, C, take this laſt remainder from the former, and the reſult will be the value ſought.

DEMONSTRATION.

Let the annuity be 1 L, r the rate of intereſt, and $a, b, c,$ *&c.* $\overset{\prime}{a}, \overset{\prime}{b}, \overset{\prime}{c},$ *&c.* the probabilities of living 1, 2, *&c.* years, as in the foregoing propoſitions. Now the expectation of C, upon what he may happen to receive at the end of any year (ſuppoſe the firſt) may be conſidered in two parts, as depending on two different events; for firſt, if C be the only perſon of the three then living, of which the probability is $c \times \overline{1-a} \times \overline{1-b}$, he will be intitled to the whole year's rent, or 1 L, therefore his preſent expectation thereon, diſcount being allowed, is $c \times \overline{1-a} \times \overline{1-b} \times \frac{1}{r}$. Secondly, he and B may happen to be living, and A only extinct, in which caſe he is to receive only $\frac{1}{2}$ of 1 L; therefore, the probability of this being $cb \times \overline{1-a}$, his expectation in this caſe will be $cb \times \overline{1-a} \times \frac{1}{2r}$: Hence, by adding theſe two values together, we have $c \times \overline{1-a} \times \overline{1-b} \times \frac{1}{r} + cb \times \overline{1-a} \times \frac{1}{2r}$ $= \frac{c}{r} - \frac{cb}{2r} - \frac{ca}{r} + \frac{abc}{2r}$ for the total expectation of C, upon what he may happen to receive at the end of the firſt year. And, from the very ſame manner of reaſoning, it will appear that $\frac{c'}{r^2} - \frac{c'b'}{2r^2} - \frac{a'c'}{r^2} + \frac{a'b'c'}{2r^2}$ is the value of the expectation of C, upon what he may happen to receive at the end of the ſecond year, *&c.* Therefore the total expectation, or preſent value of all the ſums that C may happen to receive from time to time, is

$$
\begin{aligned}
&+\frac{c}{r} - \frac{cb}{2r} - \frac{ca}{r} + \frac{abc}{2r}\\
&+\frac{c'}{r^2} - \frac{c'b'}{2r^2} - \frac{c'a'}{r^2} + \frac{a'b'c'}{2r^2}\\
&+\frac{c''}{r^3} - \frac{c''b''}{2r^3} - \frac{c''a''}{r^3} + \frac{a''b''c''}{2r^3}
\end{aligned}
$$

&c.

But $\left(\frac{c}{r} + \frac{c'}{r^2} + \frac{c''}{r^3}, \&c.\right)$ the firſt column towards the right-hand, as appears from Prob. I. expreſſes the value of an annuity upon the life C; and the ſecond column $\left(\frac{cb}{2r} + \frac{c'b'}{2r^2}, \&c.\right)$ half the value of the ſame annuity upon the two *joint* lives B C, and ſo on; whence the truth of the Solution is manifeſt.—By proceeding according to this method, the value of any reverſion, however complicated, may be determined.

PROBLEM XX.

A, B, C, *agree amongſt themſelves to purchaſe an annuity, to be equally divided between them whilſt they live together, then to be divided equally between the two ſurvivors, then to belong intirely to the laſt ſurvivor for life: To find what each perſon ought to contribute towards the purchaſe.*

SOLUTION.

From the value of the life A take half the ſum of the values of the *joint* lives A, B, and A, C, and to the remainder add $\frac{1}{3}$ of the value of the three *joint* lives A, B, C, the ſum ſhall be the value which A ought to contribute; and the like will hold with regard to the lives B and C.

DEMONSTRATION.

The expectation of A, upon what he may happen to receive at the end of one year, may be conſider'd in four parts, as depending on ſo many different events; for, firſt, A, B, and C, may be all living, the probability whereof is abc, in which caſe he is to receive $\frac{1}{3}$ of the year's rent, or $\frac{1}{3}$ of 1 L; therefore his expectation on this event is $\frac{abc}{3r}$. Secondly, A and B may be living, and C extinct, the probability whereof is $ab \times \overline{1-c}$, therefore on this event his expectation is $\frac{ab \times \overline{1-c}}{2r}$. Thirdly, A and C may be living, and B extinct, on this event his expectation is $\frac{ac \times \overline{1-b}}{2r}$. Laſtly, A may be the only perſon then living, upon this the expectation will be $\frac{a \times \overline{1-b} \times \overline{1-c}}{r}$; wherefore, by adding all theſe four values into one ſum, we have $\frac{a}{r} - \frac{ab}{2r} - \frac{ac}{2r} + \frac{abc}{3r}$ for the total value of the expectation of A, upon what he may happen to receive at the end of the firſt year; from whence, by following the ſame method of reaſoning, as in the laſt propoſition, the truth of the Solution will appear evident.

In like manner may the ſhare of A be determined, be the number of perſons concerned in the purchaſe ever ſo great, ſuppoſing the annuity to be always equally divided among the ſurviving lives; for let P expreſs the value of the life A (*viz.* the value of an annuity of 1 L for the life of A;) Q the ſum of the values of all the *joint* lives, ariſing from the combination of the life A, with each *one* of the other; R the ſum of the values of all the *joint* lives, ariſing from the combination of the life A, with each *two* of the other; S the ſum of the values of all the *joint* lives, ariſing from the combination of the life A, with each *three* of the other, and ſo on; then will $P - \frac{Q}{2} + \frac{R}{3} - \frac{S}{4} + \frac{T}{5}$, &c. be the value which A ought to contribute.

PRO-

PROBLEM XXI.

Suppoſing any given number of lives P, Q, R, *and that* A, *or his Heirs, are to receive the ſum* S *upon the firſt vacancy of any of theſe lives: To find the value of* A's *expectation in preſent money.*

SOLUTION.

Multiply the given ſum by the value of an annuity for the joint lives P, Q, R, and divide the product by the value of the ſame annuity for ever; ſubtract the quotient from the given ſum, and there will remain the value ſought.

DEMONSTRATION.

Let E be the value of an annuity for ever (*i. e.* the number of years purchaſe it is worth) and P the value of an annuity for the propoſed joint lives; therefore, ſeeing the value of the reverſion for ever, after the joint lives P, Q, R, to be received as ſoon as one of thoſe lives becomes extinct, is to the ſum (S) to be received at the ſame time, as E to S, the preſent value of that reverſion, muſt, conſequently, be to the preſent value of this ſum, in the ſame ratio of E to S; but the preſent value of the reverſion is known to be E—P, therefore that of the ſum S will be $\frac{E-P}{E} \times S = S - \frac{PS}{E}$. Q. E. D.

EXAMPLE.

Let the number of lives be 3, their ages each 27 years, the rate of intereſt 4 *per cent.* and the propoſed ſum 100 *l.* then the value of an annuity for the joint lives being (by the table) 8 years purchaſe, and the value of an annuity for ever 25 years purchaſe, we ſhall, by multiplying 100 *l.* by 8, and dividing the product by 25, have 32 *l.* which ſubtracted from 100 *l.* will leave 68 *l.* for the preſent value of 100 *l.* to be received at the firſt vacancy of the three propoſed lives.

PROBLEM XXII.

Suppoſing a given ſum S *to be depending, ſo as not to become due till the lives* P, Q, R, &c. *are all extinct; 'tis required to find the value of that ſum in preſent money.*

SOLUTION.

Multiply the given ſum by the value of an annuity for the longeſt of the propoſed lives, and divide the product by the perpetuity; ſubtract the quotient from the given ſum, and there will remain the value required.

The reaſons of this Solution will appear evident from the method laid down in the laſt Problem.

REMARK.

In what has been hitherto laid down, we have had regard to ſuch annuities as are paid yearly, but if the payments are to be made every half year, which is moſt commonly the caſe, the true value at which the purchaſe is to be eſtimated, ought to be a little increaſed; and therefore it may not be improper to conſider here, how much that increaſe will be. In order to which, let E denote the perpetuity, or the value of an annuity for ever, and P the value of the ſame annuity for any number of lives; then, ſince that life, upon whoſe failing the annuity ceaſes, has the ſame chance very nearly, to drop in the laſt as in the firſt half of the year, it is manifeſt that the purchaſer has, in this caſe, an equal chance to receive half a year's rent more than when the annuity is paid yearly; and therefore the ſum that ought to be allow'd as an equivalent, for this, to be paid upon ceaſing of the annuity, will be $\frac{1}{4}$ of a year's purchaſe, or $\frac{1}{4}$ L, of which ſum, the preſent value will, by the laſt Prob. be $\frac{E-P}{4\,E}$; but this is not the only advantage the purchaſer has in this caſe; for, beſides this, he has the uſe of the 1ſt, 3d, 5th, &c. half years rents each half a year; the value of which conſideration, ſince the intereſt of $\frac{1}{2}$ L, for half a year, is to 1 L, the whole ſum received in that year, in the conſtant proportion of 1 to 4 E, will, it is manifeſt, be to the preſent value (P) of all the ſums that may be received from time to time, in the ſame conſtant proportion; and therefore is equal to $\frac{P}{4\,E}$, which added to $\frac{E-P}{4\,E}$, above found, gives $\frac{1}{4}$ L for the whole value required. Hence it appears, that be the rate of intereſt and the number of lives what they will, the difference of the values of two equal annuities, ariſing from the rents of one being paid yearly, and the other half yearly, will, in all caſes, be $\frac{1}{4}$ of a year's purchaſe; and in like manner, it will be found, that the difference of the values of two annuities, upon account of the rents of the one being paid yearly, and the other quarterly, will, in all caſes, be $\frac{3}{8}$ of a year's purchaſe very nearly.

From the ſame method of proceeding, may the difference between the value of an annuity payable in money, and the value of an eſtate in land, where the purchaſer enjoys it, by an actual poſſeſſion, to the laſt moments of life, be determined; for, ſince it is an equal chance whether the life on which the eſtate laſt depends, drops in the former or latter half of the year, the purchaſer may be ſuppoſed, in this caſe, to enjoy an annuity half a year longer, than he who is paid in money yearly, and intirely loſes the laſt payment, if he dies but one day before it becomes due; and therefore the preſent value of this conſideration, by the laſt Prob. will be $\frac{E-P}{E} \times \frac{1}{2}$; whence it appears, that if from the perpetuity you ſubtract the value of an annuity payable in money, and divide the remainder by double the perpetuity, the reſult will be the parts of a year's purchaſe, expreſſing the difference of values required.

This is true, when the value of the eſtate is to be eſtimated from the yearly rent it would produce, or when the poſſeſſion for one whole year is exactly equivalent to 1 L, to be received at the end of that year; but if the eſtate, when let, will bring in $\frac{1}{2}$ L every half year, then there being the advantage of the intereſt of the 1ſt, 3d, 5th, &c. payments, for half a year each, the value $\frac{E-P}{2E}$, above found, ought to be increaſed by $\frac{P}{4E}$, which will bring it to $\frac{1}{2} - \frac{P}{4E}$, for the difference required here, which in moſt real caſes will be $= \frac{3}{8}$ of a year's purchaſe very near. Hence we may conclude, that if to the value of the annuity, as found in the tables, be added $\frac{1}{4}$ of a year's purchaſe, when the rents are to be paid half-yearly, or $\frac{3}{8}$ of a year's purchaſe when they are to be paid quarterly, or the purchaſe is in land; the ſum will give the value of the annuity in thoſe caſes reſpectively.

PROBLEM XXIII.

There is an eſtate, which, if A, *the preſent poſſeſſor, happens to die in a given time, or before he attains to a certain age, is afterwards to belong to* B, *and his heirs for ever; to find the value of* B's *expectation.*

SOLUTION.

The expectation of B may be conſidered in two parts, one with regard to what he may receive during the propoſed term, and the other with regard to what he may receive after that term is expired; the former of which, as all the rents that become due in that term, will belong either to him or A, muſt, it is manifeſt, be equal to the difference between the value of an annuity *certain* for (*n*) the number of years in the ſaid term, and the value of another equal annuity for the ſame number of years, as depending on the contingency of A's continuing alive to the end of that time; but the latter part of B's expectation, ſince the whole eſtate is to belong to him and his heirs for ever, if A ſhould happen to be extinct at the end of the aforeſaid term, will be the value of the perpetuity to be received n years hence, multiply'd by the probability of A's not living to the end of that time; wherefore, if this probability be denoted by P, the perpetuity by E, and the preſent value of 1 L, due n years hence, by m, and r, a, $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$, *&c.* be ſuppoſed as in the preceding propoſitions, it is evident from what has been aboveſaid, that $p\,m\,E$

$$+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}+\frac{1}{r^5}+\frac{1}{r^6}, \text{ \&c.}$$

$$-\frac{a}{r}-\frac{\overset{\prime}{a}}{r^2}-\frac{\overset{\prime\prime}{a}}{r^3}-\frac{\overset{\prime\prime\prime}{a}}{r^4}-\frac{\overset{\prime\prime\prime\prime}{a}}{r^5}-\frac{\overset{\prime\prime\prime\prime\prime}{a}}{r^6}, \text{ \&c.}$$

or,

$$p\,m\,E+\frac{1-r^{-n}}{r-1}$$

$$-\frac{a}{r}-\frac{\overset{\prime}{a}}{r^2}-\frac{\overset{\prime\prime}{a}}{r^3}-\frac{\overset{\prime\prime\prime}{a}}{r^4}, \text{ \&c.}$$

will be the total value of B's expectation; where each ſeries is to be continued to as many terms, as there are units in n. But as the finding and adding together thoſe terms, when their number is great, may ſeem to require too much labour for common practice, I have deduced the following rule therefrom; which, being applicable to the tables, gives the true anſwer without any great trouble.

RULE.

From the perpetuity ſubtract the value of a life of that age, to which the expectation of B is limited, and multiply the remainder by the probability that A lives to that age, and this product again, by the preſent value of 1 L, to be received at the end of the given term; to this laſt product add the value of the life A, in poſſeſſion, and take the ſum from the perpetuity, and there will remain the value required.

EXAMPLE.

Let the age of A be 8 years, and intereſt at 5 *per cent.* and let B be entitled to an eſtate for himſelf and heirs for ever, upon the deceaſe of A, if A ſhould happen to die before he attains to twenty-one; then the value of a life of twenty-one will be 12.9, which taken from 20, the perpetuity leaves 7.1, this multiply'd by $\frac{455}{541}$, the probability of a perſon of the age of 8 living to 21, gives 5.97, which multiply'd by 0.53, the preſent value of 1 L to be received *n* years hence, will be 3.16, this added to 14.3, the value of the life A, gives 17.46, which ſum, deducted from the perpetuity, leaves 2.54, for the value of B's expectation, which is little more than $2\frac{1}{2}$ year's purchaſe.

PROBLEM XXIV.

Q *expects an eſtate for himſelf and heirs for ever, after the extinction of a given number of lives* A, B, C, *provided this happens in a given number* (n) *of years; to find at how much the value of* Q*'s expectation is to be eſtimated.*

SOLUTION.

Let every thing be as in the preceding problem, only let P here denote the probability that all the lives A, B, C, *&c.* drop in *n* years; then by following the very ſame method of reaſoning, there laid down, it will appear that pmE

$$+\frac{\overline{1-a}\times\overline{1-b}\times\overline{1-c}}{r}, \&c. +\frac{\overline{1-a'}\times\overline{1-b'}\times\overline{1-c'}}{r^2}, \&c.$$

$$+\frac{\overline{1-a''}\times\overline{1-b''}\times\overline{1-c''}}{r^3}, \&c.$$ will be the true va-

lue of B's expectation; where the ſeries is to be continued to *n* terms.

Note, the foregoing ſeries, in caſe of two or three lives, might be referred to the tables, as that in the laſt Problem was; but the Advantage gained thereby, in computation, would not be ſo conſiderable.

OF

SUCCESSIVE LIVES.

PROBLEM XXV.

SUppoſing A *to enjoy an annuity for life, and at his deceaſe to have the nomination of a Succeſſor, who is likewiſe to enjoy the annuity for his life; to find the preſent value of the two Succeſſive Lives.*

SOLUTION.

Let the value of the life A, in poſſeſſion, be denoted by P; and let the life to be put in nomination, at the deceaſe of A, be ſuch, that the value of an annuity granted thereon, may be equal to Q, and let D be the required value of the two lives, and E the value of the annuity for ever, and let $r, a, \overset{\prime}{a}, \overset{\prime\prime}{a}$, *&c.* be as in the preceding propoſitions. Therefore, ſince the probability that the firſt life fails, or that the ſecond comes into poſſeſſion, the firſt year is $1-a$; and as the total value of what the ſecond life will be entitled to at the happening of this event, is Q, the expectation on the ſecond, upon the contingency of coming into poſſeſſion the firſt year, will, it is evident, be $\overline{1-a}\times Q$. Moreover, becauſe the probability that the ſecond life comes into poſſeſſion the ſecond year is $a-\overset{\prime}{a}$, the expectation on the whole value of the annuity, upon the probability of entering into poſſeſſion the ſecond year, (diſcount for one year being allow'd) will be $\frac{\overline{a-\overset{\prime}{a}}}{r}\times Q$. And after the ſame manner will the expectations, upon the 3d, 4th, 5th, *&c.* years, appear to be $\frac{\overline{\overset{\prime}{a}-\overset{\prime\prime}{a}}\times Q}{r^2}$, $\frac{\overline{\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a}}\times Q}{r^3}$, $\frac{\overline{\overset{\prime\prime\prime}{a}-\overset{\prime\prime\prime\prime}{a}}\times Q}{r^4}$, *&c.* reſpectively; therefore the ſum of all theſe, or $Q\times\overline{1+\frac{a}{r}+\frac{\overset{\prime}{a}}{r^2}+\frac{\overset{\prime\prime}{a}}{r^3}+\frac{\overset{\prime\prime\prime}{a}}{r^4}}$, *&c.* $-Q\times\overline{a+\frac{\overset{\prime}{a}}{r}+\frac{\overset{\prime\prime}{a}}{r^2}+\frac{\overset{\prime\prime\prime}{a}}{r^3}+\frac{\overset{\prime\prime\prime\prime}{a}}{r^4}}$, *&c.* muſt be equal to (D—P) the *preſent, total,* value

lue of the ſecond life. But $\frac{a}{r} + \frac{\overset{\prime}{a}}{r^2} + \frac{\overset{\prime\prime}{a}}{r^3}$, &c. is the value of the life A; wherefore, by writing P inſtead of $\frac{a}{r} + \frac{\overset{\prime}{a}}{r^2}$, &c. we have $Q \times \overline{1 + P} - Q \times rP = D - P$, and conſequently $D = P + Q - \overline{r - 1} \times PQ = P + Q - \frac{PQ}{E}$; whence it appears, that if from the ſum of the values of the two ſingle lives be taken their product, divided by the perpetuity, the remainder will be the value of the two lives in ſucceſſion. Q. E. I.

Otherwiſe,

Since the value of the ſecond life in ſucceſſion, to be received at the deceaſe of A, is to the value of the reverſion for ever, to be received at the ſame time, in the ratio of Q to E, the preſent value of the former of theſe will be to the preſent value of the latter in the ſame ratio; but the preſent value of the latter, or of the reverſion for ever after the life of A, is given equal to E—P; therefore the preſent value of the former, will be equal to $\frac{Q}{E} \times \overline{E - P} = Q \times \overline{1 - \frac{P}{E}}$, which being added to P, the preſent value of the life A, gives

P

$P + Q - \frac{PQ}{E}$, the very ſame as was before determined.

EXAMPLE.

Let the preſent value of the life A, in poſſeſſion, be equal to 11.2 years purchaſe, and let the life put in nomination at the deceaſe of A, be worth 16.3 years purchaſe, and the value of an annuity for ever, equal to 25 years purchaſe; then the ſum of the values of the two lives will be 27.5, and their product 182.56; this laſt divided by 25, the value of the perpetuity, gives 7.3, which being ſubtracted from 27.5, there will remain 20.2, for the value of the two Succeſſive Lives.

PROBLEM XXVI.

Three lives A, B, C, *being given in ſucceſſion; to find their preſent value.*

SOLUTION.

Let the values of the three lives, conſidered as independent of each other, be reſpectively denoted by P, Q, R, and let E be the value of an annuity for ever. Therefore, ſince the preſent value of the two firſt lives in ſucceſſion (A and B) is, by the laſt Problem, equal to $P + Q - \frac{PQ}{E}$, the value of the reverſion of the annuity for ever, after theſe two lives, will be $E - P - Q + \frac{PQ}{E}$; and therefore, from the ſame method of reaſoning as in the preceding Problem, it will be as $E : R :: E - P - Q + \frac{PQ}{E}$ to $R - \frac{PR - RQ}{E} + \frac{PQR}{EE}$, or $R \times \overline{1 - \frac{P}{E}} \times \overline{1 - \frac{Q}{E}}$, the preſent value of the third life; this therefore added to $P + Q - \frac{PQ}{E}$, the value of the two firſt, gives $P + Q + R - \frac{PQ - PR - QR}{E} + \frac{PQR}{EE}$

$$= E \text{ into } 1 - \overline{1 - \frac{P}{E}} \times \overline{1 - \frac{Q}{E}} \times \overline{1 - \frac{R}{E}}$$

for the required value of the three ſucceſſive lives; from whence the value of any number of lives in ſucceſſion, may be derived by inſpection; and thence the following general Rule.

Multiply the value of each of the propoſed lives by the intereſt of 1 L for one year, taking the ſeveral products from unity, and multiplying together all the remainders, let the product thus ariſing be alſo taken from unity, and the remainder multiply'd into the value of the annuity for ever; then will the reſult be the value of an annuity for all the ſucceſſive lives.

EXAMPLE.

Let there be three lives given in ſucceſſion, whoſe values ſeparately conſider'd, are reſpectively equal to 8, 10, and 15 years purchaſe, and let intereſt be at 4 *per cent.* then the ſeveral given values, multiply'd by 0.04, will be 0.32, 0.4, and 0.6; theſe ſeverally ſubtracted from unity, leave 0.68, 0.6, and 0.4, whoſe product 0.1632 being taken from unity, there will remain 0.8368, and this multiply'd by 25, the perpetuity, gives 20.92 for the preſent value of the three lives.

PROBLEM XXVII.

Suppoſing A *to purchaſe an eſtate, in copyhold, upon any number of lives* P, Q, R, S, *for the ſum* b, *on condition that he and his heirs ſhall renew it continually, whenever any life becomes vacant, for the ſum* c; *to find the preſent value of the whole purchaſe allow'd for that eſtate.*

SOLUTION.

Let the values of the lives P, Q, R, *&c.* upon which the leaſe is firſt granted, be p, q, r, *&c.* and e the perpetuity, and let the values of the lives ($\overset{\prime}{P}$, $\overset{\prime\prime}{P}$, $\overset{\prime\prime\prime}{P}$, *&c.*) which follow in a direct ſucceſſion from P, be denoted by $\overset{\prime}{p}$, $\overset{\prime\prime}{p}$, $\overset{\prime\prime\prime}{p}$, *&c.* and thoſe of the lives ($\overset{\prime}{Q}$, $\overset{\prime\prime}{Q}$, $\overset{\prime\prime\prime}{Q}$, *&c.*) following in a direct ſucceſſion from Q, by $\overset{\prime}{q}$, $\overset{\prime\prime}{q}$, $\overset{\prime\prime\prime}{q}$, *&c.* reſpectively, and ſo on, with regard to the reſt of the lives R, S, T; then it will be as $\overset{\prime}{p} : c :: \overset{\prime}{p} \times \overline{1 - \frac{p}{e}}$, the preſent value of the life $\overset{\prime}{P}$ (by Prob. XXV.) to $c \times \overline{1 - \frac{p}{e}}$ the preſent value of the ſum (c) to be paid at the deceaſe of P, or the nomination of the new life $\overset{\prime}{P}$; and as $\overset{\prime\prime}{p} : c :: \overset{\prime\prime}{p} \times \overline{1 - \frac{p}{e}} \times \overline{1 - \frac{\overset{\prime}{p}}{e}}$ the preſent value of the life $\overset{\prime\prime}{P}$ (by Prob. XXVI.) to $c \times \overline{1 - \frac{p}{e}} \times \overline{1 - \frac{\overset{\prime}{p}}{e}}$ the preſent value of the ſum (c) to be paid upon the deceaſe of $\overset{\prime}{P}$, or the nomination of $\overset{\prime\prime}{P}$, the next life in this ſucceſſion: And in the ſame manner will the preſent values of the ſums to be paid at the nomination of the lives $\overset{''}{P}$, $\overset{'''}{P}$, &c. be found $c \times \overline{1 - \frac{p}{e}} \times \overline{1 - \frac{p'}{e}} \times \overline{1 - \frac{p''}{e}}$, and $c \times \overline{1 - \frac{p}{e}} \times \overline{1 - \frac{p'}{e}} \times \overline{1 - \frac{p''}{e}} \times \overline{1 - \frac{p'''}{e}}$, &c. But the ſum of all theſe, or $c \times \overline{1 - \frac{p}{e}} + c \times \overline{1 - \frac{p}{e}} \times \overline{1 - \frac{p'}{e}} + c \times \overline{1 - \frac{p}{e}} \times \overline{1 - \frac{p'}{e}} \times \overline{1 - \frac{p''}{e}}$, &c. continued *in infinitum*, is the preſent value of all the ſums that may be paid from time to time, for the renewals of all the lives in this ſucceſſion; and from the ſame way of reaſoning, the preſent value of all the ſums, that may be paid for the renewals of all the lives in the ſucceſſions $\overset{'}{Q}$, $\overset{''}{Q}$, $\overset{'''}{Q}$, &c. $\overset{'}{R}$, $\overset{''}{R}$, $\overset{'''}{R}$, &c. &c. will appear to be $c \times \overline{1 - \frac{q}{e}} + c \times \overline{1 - \frac{q}{e}} \times \overline{1 - \frac{q'}{e}} + c \times \overline{1 - \frac{q}{e}} \times \overline{1 - \frac{q'}{e}} \times \overline{1 - \frac{q''}{e}}$, &c. and $c \times \overline{1 - \frac{r}{e}} + c \times \overline{1 - \frac{r}{e}} \times \overline{1 - \frac{r'}{e}} + c \times \overline{1 - \frac{r}{e}} \times \overline{1 - \frac{r'}{e}} \times \overline{1 - \frac{r''}{e}}$, &c. &c. reſpectively.

Therefore, if to the ſum of all theſe, the value b, paid at entring be added, the aggregate,

$$b + c \times \overline{\overline{1-\frac{p}{e}} + \overline{1-\frac{p}{e}} \times \overline{1-\frac{p'}{e}} + \overline{1-\frac{p}{e}} \times \overline{1-\frac{p'}{e}} \times \overline{1-\frac{p''}{e}}, \&c.} + c \times \overline{\overline{1-\frac{q}{e}} + \overline{1-\frac{q}{e}} \times \overline{1-\frac{q'}{e}} + \overline{1-\frac{q}{e}} \times \overline{1-\frac{q'}{e}} \times \overline{1-\frac{q''}{e}}, \&c.} + c \times \overline{\overline{1-\frac{r}{e}} + \overline{1-\frac{r}{e}} \times \overline{1-\frac{r'}{e}} + \overline{1-\frac{r}{e}} \times \overline{1-\frac{r'}{e}} \times \overline{1-\frac{r''}{e}}, \&c.}\ \&c.$$

will, it is evident, be the preſent total value of all that A and his heirs may pay for the enjoyment of the eſtate for ever. Q. E. I.

COROLLARY I.

Hence, if the lives with which the leaſe is filled up from time to time, be equal to one another, the general expreſſion will become

$$b + c \times \overline{1-\frac{p}{e}} \times \overline{1 + \overline{1-\frac{p'}{e}}\Big|^{1} + \overline{1-\frac{p'}{e}}\Big|^{2} + \overline{1-\frac{p'}{e}}\Big|^{3}, \&c.} +$$

$+\overline{c\times 1-\frac{q}{e}}\times\overline{1+\overline{1-\frac{p'}{e}}^{1}+\overline{1-\frac{p'}{e}}^{2},\ \&c.}$

which therefore, becauſe $1+\overline{1+\frac{p'}{e}}^{1}$ $+\overline{1-\frac{p'}{e}}^{2}+\overline{1-\frac{p'}{e}}^{3}+\overline{1-\frac{p'}{e}}^{4}$, &c. is equal to $\frac{e}{p'}$, will become $b+\frac{c}{p'}\times\overline{e-p}$ $\overline{+e-q+e-r,\ \&c.}$

COROLLARY II.

But if all the lives, as well thoſe upon which the leaſe is firſt granted, as thoſe put into nomination afterwards, be equal to each other, the value allowed for the whole purchaſe will be $b+\frac{nc\times\overline{e-p}}{p}$.

COROLLARY III.

Laſtly, if $b+c\times\overline{\overline{1-\frac{p}{e}}+\overline{1-\frac{p}{e}}\times\overline{\overline{1-\frac{p'}{e}}}},$ $\overline{\&c.+c\times\overline{1-\frac{q}{e}}+\overline{1-\frac{q}{e}}\times\overline{1-\frac{q'}{e}}},\ \&c.$ &c. be taken $=e$, we ſhall have $c=$
$e-b$

$$\frac{e-b}{1-\frac{p}{e}+\overline{1-\frac{p}{e}}\times\overline{1-\frac{p'}{e}},\ \&c.+1-\frac{q}{e}+\overline{1-\frac{q}{e}}\times\overline{1-\frac{q'}{e}}\ \&c.}$$

ſhewing the true value that A and his heirs ought in juſtice to pay at each renewal; which value, when the lives renewed with are all equal, will therefore be

$$\frac{\overline{e-b}\times p'}{e-p+e-q+e-r+e-s,\ \&c.}$$

The above Corollaries, expreſſed in words at length, afford the following practical Rules.

I. Subtract the ſum of the values of all the lives, upon which the leaſe is firſt granted, from the perpetuity multiply'd by the number of thoſe lives, and divide the remainder by the value of one of the equal lives, with which the leaſe is from time to time to be renewed, and multiply the quotient by the ſum agreed upon to be paid for renewing, and the product will be the preſent value of all the ſums that may be paid for all the renewals for ever; which added to the value paid at firſt entring will give the total value of the purchaſe.

II. Multiply the rent of one year by the preſent value of all the ſums that may be paid for renewals (found as above) and divide the product by the perpetuity, and the quotient will be the ſum by which the rent-roll of the firſt proprietor's eſtate, ought to be increaſed upon account of ſuch renewals.

III. Take the difference between the value paid upon firſt entering, and the perpetuity, and multiply it by the value of one of the equal lives, with which the leaſe is to be conſtantly renewed, divide the product by the exceſs of the rectangle of the perpetuity into the given number of lives, above the ſum of the values of all thoſe lives, and the quotient will be the ſum, which, in juſtice, ought to be conſtantly paid for renewing.

EXAMPLE.

Let the propoſed eſtate be 100 *l.* *per annum*, and the number of lives upon which the leaſe is granted be 3, and let their values, ſeparately conſidered, be worth 10, 12, and 15 years purchaſe; let the ſum paid upon entring be 1600 *l.* and that for renewing 400 *l.* and intereſt at 4 *per cent.* and ſuppoſe the purchaſer to have the liberty of renewing with lives of what ages he ſhall think proper, or moſt to his own advantage.

Then the ſum of the values of the three lives will be 37 years purchaſe, which taken from 75, three times the perpetuity, leaves 38 years purchaſe; but the greateſt value of an annuity for one ſingle life (at 4 *per cent.*) is, by the table 16.4; wherefore, dividing 38 by 16.4, we have 2.317, which multiply'd by 4, the number of years purchaſe paid for each renewal, gives 9.268 years purchaſe, or 926*l.* 16*s.* for the preſent value of all the ſums that may be paid from time to time for renewing; therefore 2526*l.* 16*s.* is the whole value of the purchaſe; from whence it will be found that the ſum by which the rent-roll of the eſtate of the firſt proprietor, ought to be increaſed upon account of thoſe renewals, is 37*l.* 1*s.* and that 388*l.* 6*s.* is the ſum that ought to be paid, in juſtice, at each renewal.

OF

REVERSIONS,

WHERE THE

Expectation depends upon the Probability of one particular Life, in poſſeſſion, ſurviving the reſt.

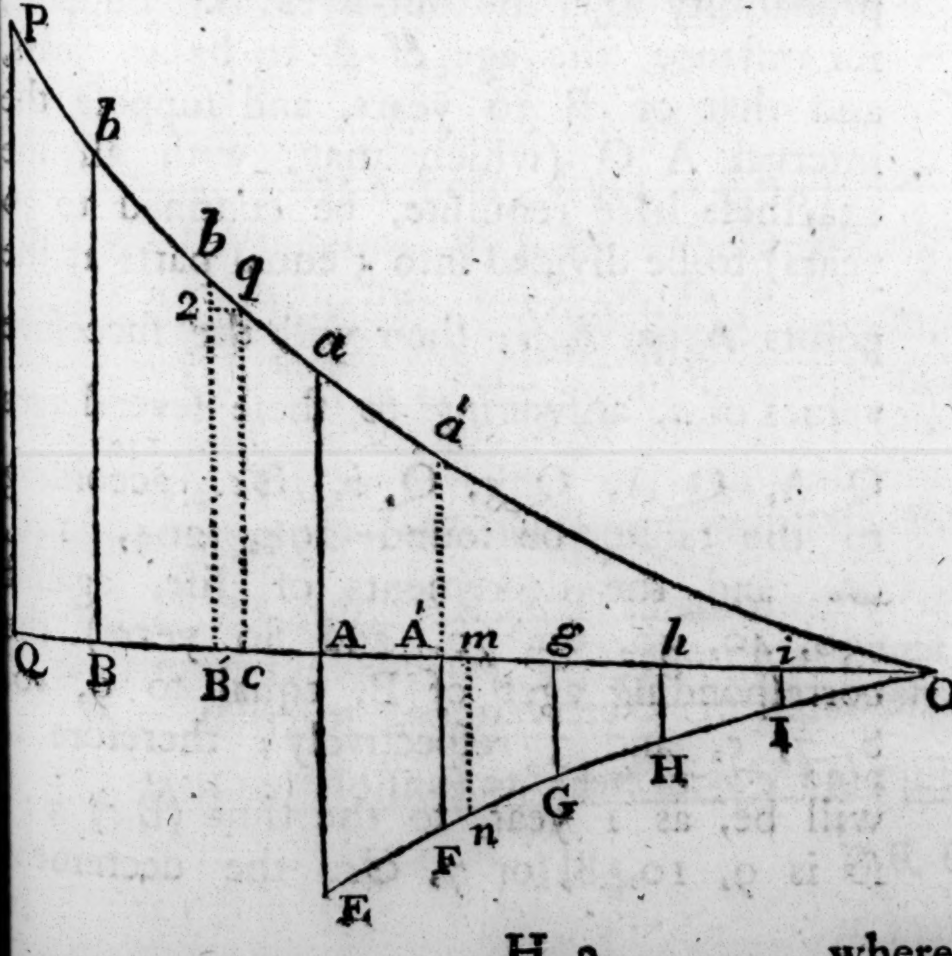
LEMMA II.

THE ages of two perſons A and B *being given; to determine from the table of Obſervations, the probability which each of them has to ſurvive the other.*

Let QO repreſent the whole extent of life, continued from the inſtant of birth at Q, to the extremity of old age at O, and let QA, and QB, be the given ages of A and B, and QA' and QB' any other two correſponding ages to which they have a chance of attaining; and let P*ba*O be a curve, whoſe ordinates B*b*, $B'b'$, A*a*, *&c.* repreſent the numbers of perſons anſwering, in the table of Obſervations, to thoſe ages; ſuppoſe *cq* as near as poſſible to $B'b'$, and

$A'm$

$\acute{A}m$ equal to $\acute{B}c$, and $q2$ parallel to QO: put $Aa=a$, $\acute{A}\acute{a}=\acute{a}$, $Bb=b$, $b2=\dot{b}$, and $q2=\acute{A}m=\dot{x}$; and ſuppoſe A, if he be the ſurvivor, to receive the ſum S. Then will the probability of his receiving that ſum, during the interval $\acute{A}m$, be compounded of the probability of his attaining to the age $Q\acute{A}$, and that of B's dying in the correſponding interval $\acute{B}c$, of which the former being $\frac{\acute{a}}{a}$, and the latter $\frac{\dot{b}}{b}$, the compound of both muſt conſequently be $\frac{\acute{a}\dot{b}}{ab}$. Let this value, therefore, be defined (every



where) by an area $\overset{\prime}{A}mnF\overset{\prime}{A}$, ſuppoſing $EFnO$ to be a curve, whoſe whole area $AOEA$ expreſſes the probability that A receives the ſum S ſome time during the whole interval AO; then by dividing $\frac{\overset{\prime}{a}\dot{b}}{ab}$ by $\dot{x}$ $(=\overset{\prime}{A}m)$ we ſhall have $\overset{\prime}{A}F = \frac{\overset{\prime}{a}\dot{b}}{ab\dot{x}}$ for the general equation of this curve; from which, by a known method for approximating the areas of curves, by means of equidiſtant ordinates, may the area $AOEA$ be determined; which area, as it expreſſes the probability that A lives to receive the ſum S, muſt alſo expreſs the probability that he out-lives B. Suppoſe, for inſtance, the age of A to be 40 years, and that of B 30 years, and ſuppoſe the interval AO (which may, with all the exactneſs here requiſite, be confined to 50 years) to be divided into 5 equal parts at the points $\overset{\prime}{A}$, g, h, i; then will the ſucceſſive values of $\overset{\prime}{a}$, anſwering to theſe ſeveral ages QA, $Q\overset{\prime}{A}$, Qg, Qh, &c. according to the table, be found 294, 204, 130, &c. and the decrements of life, againſt (30, 40, 50, 60, 70, and 80 years) the correſponding ages of B, equal to 9, 10, 8, 7, 5, and 3 reſpectively; therefore it will be, as 1 year, to the time $(\overset{\prime}{B}c)$ $\dot{x}$ :: ſo is 9, 10, 8, or 7, &c. the decrement in one year, to 9 $\dot{x}$, 10 $\dot{x}$, 8 $\dot{x}$, or 7 $\dot{x}$, &c. the decrement in the time $\dot{x}$ very near; which values, therefore, with thoſe above, being ſucceſſively wrote for $\dot{b}$ and $\acute{a}$, in the general expreſſion $\frac{\acute{a}\,\dot{b}}{\dot{x}\,a\,b}$, the ordinates A E, ÁF, *g* G, &c. will, in this caſe, come out $\frac{294 \times 9}{294 \times 385}$, $\frac{204 \times 10}{294 \times 385}$, $\frac{130 \times 8}{294 \times 385}$, $\frac{69 \times 7}{294 \times 385}$, $\frac{29 \times 5}{294 \times 385}$, and $\frac{0 \times 3}{294 \times 385}$ reſpectively: therefore the value of the mean ordinate will here be $\frac{290299}{294 \times 385 \times 288} = 0.008905$, which multiplied by A O = 50, gives 0.44525 for the required probability of A ſurviving B; whence the probability of the contrary, that B ſurvives A, will be $1 - 0.44525 = 0.55475$.

Note. That a mean, from a given number of ordinates, may be had by the following theorems, where *a*, *b*, *c*, &c. denote the given ordinates ranged according to order, and *m* is the mean ordinate correſponding.

3. $\frac{a + 4b + c}{6} = m$ 4. $\frac{a + 3b + 3c + d}{8} = m$

5. $\frac{7a + 32b + 12c + 32d + 7e}{90} = m$

6. $\frac{19a + 75b + 50c + 50d + 75e + 19f}{288} = m$

7. $\frac{41a + 216b + 27c + 272d + 27e + 216f + 41g}{840} = m$

PROBLEM XXVIII.

A and B are in joint poſſeſſion of an annuity; which, if A be the longer liver, is, after both lives are extinct, to belong to C and his heirs for ever: To find the preſent value of the expectation of C, on that annuity.

SOLUTION.

From the value of the annuity for ever, ſubtract the value of the two lives in poſſeſſion, and multiply the remainder by the probability of A ſurviving B (found by the preceding Lemma) and the product will be the value required.

For ſince the value of the reverſion which C and his heirs expect, did it not depend on A's ſurviving B, would be the exceſs of the value of the annuity for ever, above the value of the two lives A and B, that exceſs multiply'd by the probability of A ſurviving, muſt conſequently be the true value of the reverſion, allowing for that contingency.

EXAMPLE.

Let the age of A be 40, that of B 30, and intereſt of money at 4 *per cent.* then the value of the two lives A and B will be 15.8 years purchaſe (by Caſe II. Prob. X.) which taken from 25 years purchaſe, the value of the annuity for ever, there will remain 9.2, and this multiply'd by 0.445, the probability of A ſurviving B, gives 4.1 for the number of years purchaſe required.

PROBLEM XXIX.

There are three perſons A, B, C, *the two former of whom* A, B, *enjoy an annuity between them, which annuity, if* A *ſurvives* B, *is afterwards to belong equally to* A *and* C, *during their joint continuance, and then intirely to* C *for life, if he be the laſt ſurvivor; to find the value of* C's *expectation on that annuity.*

SOLUTION.

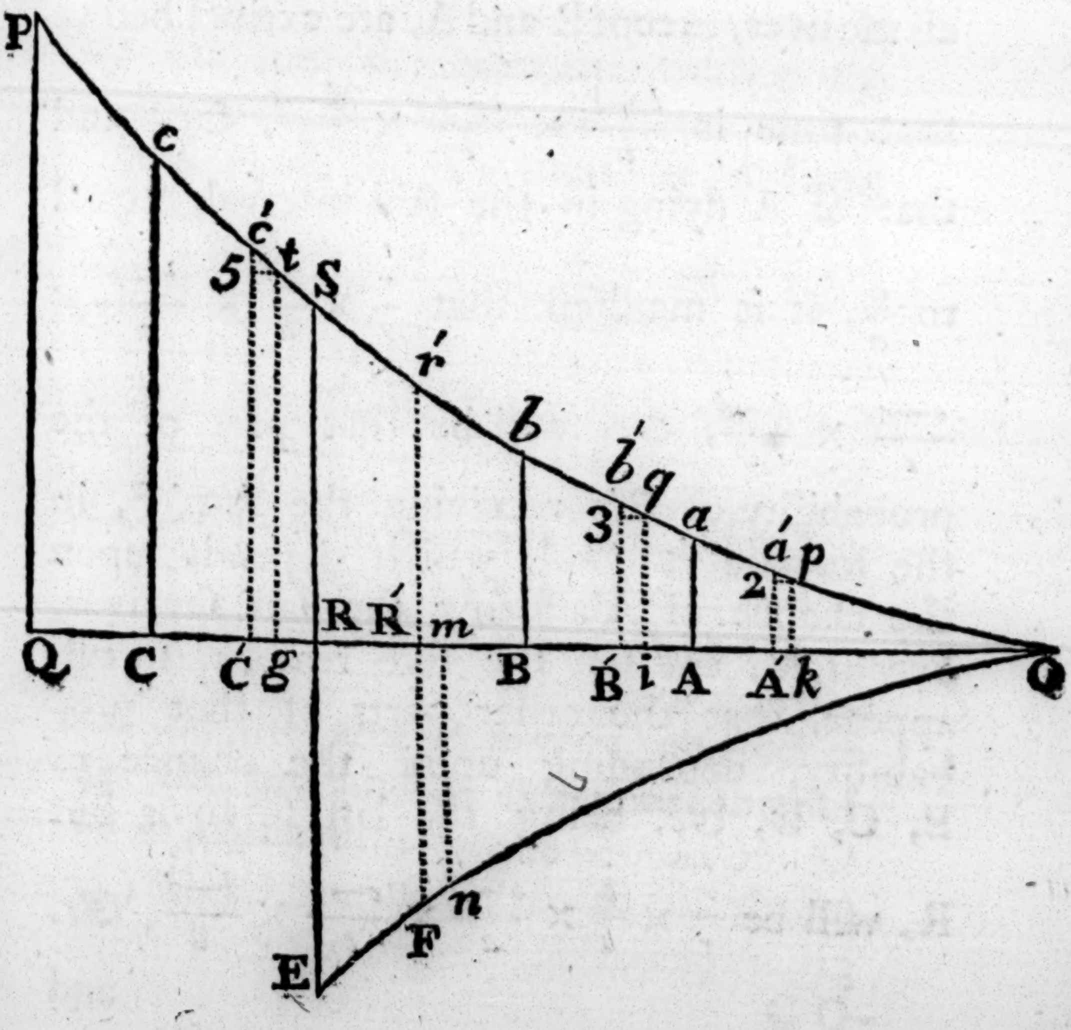
Take the exceſs of the value of the three lives A, B, C, above the value of the two lives A, B, and alſo the exceſs of the value of the two *joint* lives A, C, above that of the three *joint* lives A, B, C; multiply the former of them by the probability that A ſurvives B, and to the product add half the latter, and the ſum thus ariſing will be the true value of C's expectation.

For ſince the firſt of the abovementioned exceſſes would be the preſent value of all the rents that C might receive after the deceaſe of A and B, was not that value to depend on A's ſurviving B, the true value of thoſe rents, allowing for that contingency, muſt be the product of that exceſs, by the probability of A ſurviving B: therefore, as the latter of thoſe exceſſes is (by Prob. V.) double the value of all the other rents that he may receive, during the joint continuance of himſelf and A, the ſolution is manifeſt.

LEMMA III.

The ages of any number of perſons R, A, B, C, D, *&c. being given; to find the probability that any one of them, pitch'd upon, as* R, *ſhall ſurvive the reſt.*

Let QO repreſent the whole term of life, continued from the time of birth at Q, to the extremity of old age at O; and let QR, QA, QB, QC, *&c.* be the given ages of R, A, B, C, *&c.* reſpectively, ly, and QŔ, QÁ, QB́, QĆ, *&c.* any other contemporary ages to which thoſe perſons have a chance of living; let PSO be a curve whoſe ordinates C*c*, RS, B*b*, *&c.* every where expreſs the numbers of perſons, in the table of Obſervations, anſwering to the correſponding ages QC, QR, QB, *&c.* let *kp* be as near as can be to Á*á*, and Á*k*, B́*i*, Ŕ*m*, Ć*g*, *&c.* all equal to one another, and let *p* 2, *q* 3, and *t* 5, *&c.* be parallel to QO. Put RS



$= r$, $R\acute{r} = \acute{r}$, $\acute{R}m = x$, $Aa = a$, $\acute{A}\acute{a} = \acute{a}$, $a2 = \dot{a}$, $Bb = b$, $\acute{B}\acute{b} = \acute{b}$, $\acute{b}3 = \dot{b}$, $Cc = c$, $\acute{C}\acute{c} = \acute{c}$, $\acute{c}5 = \dot{c}$, *&c.* and let R, in caſe he be the laſt ſurvivor, be intitled to receive the ſum S; then will the probability of his receiving that ſum, in the interval $\acute{R}m$, depend upon theſe events; firſt, that he attains to the age $Q\acute{R}$; ſecondly, that only one of the other perſons ſhall then remain alive; and laſtly, that this perſon ſhall alſo go out of being, during that interval; wherefore, ſince the probability that all the lives, except R and A, are extinct before that time is $\frac{b-\acute{b}}{b} \times \frac{c-\acute{c}}{c} \times \frac{d-\acute{d}}{d}$, *&c.* and that of A dying in the ſaid interval, equal to $\frac{\dot{a}}{a}$, it is manifeſt that $\frac{\acute{r}}{r} \times \frac{\dot{a}}{a} \times \frac{b-\acute{b}}{b} \times \frac{c-\acute{c}}{c} \times \frac{d-\acute{d}}{d}$, *&c.* will be that part of the probability of R's receiving the ſum S, in the foreſaid interval, which depends upon the chance of A's being the laſt ſurvivor, except R; and for the like reaſons it will appear, that the other parts of that probability, depending upon the chance of B, C, D, *&c.* being the laſt ſurvivor but R, will be $\frac{\acute{r}}{r} \times \frac{\dot{b}}{b} \times \frac{a-\acute{a}}{a} \times \frac{c-\acute{c}}{c} \times \frac{d-\acute{d}}{d}$, *&c.*

and $\frac{r'}{r} \times \frac{\dot{c}}{c} \times \frac{a-a'}{a} \times \frac{b-b'}{b} \times \frac{d-d'}{d}$, *&c. &c.* reſpectively; therefore the ſum of all theſe, or its equal $p \times \frac{\dot{a}}{a-a'} + \frac{\dot{b}}{b-b'} + \frac{\dot{c}}{c-c'} + \frac{\dot{d}}{d-d'}$, *&c.* by putting $p = \frac{r'}{r} \times \frac{a-a'}{a} \times \frac{b-b'}{b} \times \frac{c-c'}{c}$, *&c.* will be the whole probability of receiving the ſaid ſum during that interval. Let this probability be, every where, repreſented by the area $R' m n F R'$, ſuppoſing E F O to be a curve, whoſe whole area R E O R expreſſes the probability that R receives the ſum S, ſometime during the whole interval R O, or ſurvives all the other perſons A, B, C; then by dividing $\overline{p \times \frac{\dot{a}}{a-a'} + \frac{\dot{b}}{b-b'} + \frac{\dot{c}}{c-c'} + \frac{\dot{d}}{d-d'}}$, *&c.* by $\dot{x}$ ($= R\,m$) we ſhall have $R\,F = \frac{p}{\dot{x}} \times \overline{\frac{\dot{a}}{a-a'} + \frac{\dot{b}}{b-b'} + \frac{\dot{c}}{c-c'} + \frac{\dot{d}}{d-d'}}$, *&c.* for the general equation of this curve; from which, by the method and theorems laid down in Lemma II. the area R E O R, and conſequently the probability expreſſed thereby, may be determined.

COROLLARY.

If the probability of life be conſidered as decreaſing uniformly, then will P S O become a right-line, and $a-a'$, $b-b'$, $c-c'$, &c. as alſo $\dot{a}$, $\dot{b}$, $\dot{c}$, &c. equal to each other; therefore, by putting $a-a'=z$, $\dot{a}=\dot{z}$, and n the whole number of perſons given, the probability of receiving the ſum S in the interval $R'm$, will be defined by $\frac{n-1}{rabcd,\ \&c.}\times rz^{n-2}\dot{z}-z^{n-1}\dot{z}$; whoſe fluent (after proper correction) will, when $x=r$, be $\frac{1}{rabcd,\ \&c.}$ multiply'd into the two following ſeries $ra^{n-1}+ra\times\overline{b^{n-2}-a^{n-2}}+rab\times\overline{c^{n-3}-b^{n-3}}+rabc\times\overline{d^{n-4}-c^{n-4}}$ &c. $-\frac{\overline{n-1}\times a^{n}}{n}-\frac{\overline{n-2}\times a\times\overline{b^{n-1}-a^{n-1}}}{n-1}$ $-\frac{\overline{n-3}\times ab\times\overline{c^{n-2}-b^{n-2}}}{n-2}$ $-\frac{\overline{n-4}\times abc\times\overline{d^{n-3}-c^{n-3}}}{n-3}$, &c. where each ſeries is to be continued to as many terms, as will expreſs the place which the perſon propoſed obtains, reckoning according to ſeniority, provided that perſon be not the youngeſt of all, in which caſe, one term leſs than that number is to be taken. But the fluent thus found, is the required probability of ſurvivorſhip, according to the above hypotheſis, and, with a little correction, will ſerve ſufficiently near in real caſes.

PROBLEM XXX.

To determine the value of a reverſion for ever, after any number of lives A, B, C, *&c. upon the contingency of one particular life* A *being the laſt ſurvivor.*

SOLUTION.

From the value of the annuity for ever ſubtract the value of all the given lives, and multiply the remainder by the probability that A is the laſt ſurvivor (found by the above Lemma) and the product will be the value required; which will appear manifeſt from the reaſons already laid down in the two preceding problems.

A TABLE *ſhewing the preſent value of one pound, to be received at the end of any number of years, not exceeding 90, diſcounting at the rates of* 5, 4, *and* 3 per cent. *compound intereſt.*

Years.	Value at 5 *per cent.*	Value at 4 *per cent.*	Value at 3 *per cent.*	Years.	Value at 5 *per cent.*	Value at 4 *per cent.*	Value at 3 *per cent.*
1	.9524	.9615	.9709	21	.3589	.4388	.5375
2	.9070	.9245	.9426	22	.3418	.4219	.5219
3	.8638	.8890	.9151	23	.3255	.4057	.5067
4	.8227	.8548	.8885	24	.3100	.3901	.4919
5	.7835	.8219	.8626	25	.2953	.3757	.4776
6	.7462	.7903	.8375	26	.2812	.3607	.4637
7	.7107	.7599	.8131	27	.2678	.3468	.4502
8	.6768	.7307	.7894	28	.2551	.3335	.4371
9	.6446	.7026	.7664	29	.2429	.3206	.4243
10	.6139	.6756	.7441	30	.2314	.3083	.4120
11	.5847	.6496	.7224	31	.2204	.2965	.4000
12	.5568	.6246	.7014	32	.2099	.2851	.3883
13	.5303	.6006	.6809	33	.1999	.2741	.3770
14	.5051	.5775	.6611	34	.1903	.2636	.3660
15	.4810	.5553	.6419	35	.1813	.2534	.3554
16	.4581	.5339	.6232	36	.1726	.2437	.3450
17	.4363	.5134	.6050	37	.1644	.2343	.3350
18	.4155	.4936	.5874	38	.1566	.2253	.3252
19	.3957	.4746	.5703	39	.1491	.2166	.3158
20	.3769	.4564	.5537	40	.1420	.2083	.3066

Years.	Value at 5 *per cent.*	Value at 4 *per cent.*	Value at 3 *per cent.*	Years.	Value at 5 *per cent.*	Value at 4 *per cent.*	Value at 3 *per cent.*
41	.1353	.2003	.2977	66	.0399	.0751	.1421
42	.1288	.1926	.2890	67	.0380	.0722	.1380
43	.1227	.1852	.2805	68	.0362	.0695	.1340
44	.1169	.1780	.2724	69	.0345	.0668	.1301
45	.1113	.1712	.2644	70	.0329	.0642	.1263
46	.1060	.1646	.2567	71	.0313	.0617	.1226
47	.1010	.1583	.2493	72	.0298	.0594	.1190
48	.0961	.1522	.2420	73	.0284	.0571	.1156
49	.0916	.1463	.2349	74	.0270	.0549	.1122
50	.0872	.1407	.2281	75	.0257	.0528	.1089
51	.0830	.1353	.2215	76	.0245	.0508	.1058
52	.0791	.1301	.2150	77	.0233	.0488	.1027
53	.0753	.1251	.2087	78	.0222	.0469	.0997
54	.0717	.1203	.2027	79	.0212	.0451	.0968
55	.0683	.1156	.1968	80	.0202	.0434	.0940
56	.0651	.1112	.1910	81	.0192	.0417	.0912
57	.0620	.1069	.1855	82	.0183	.0401	.0886
58	.0590	.1028	.1801	83	.0174	.0386	.0860
59	.0562	.0989	.1748	84	.0166	.0371	.0835
60	.0535	.0951	.1697	85	.0158	.0357	.0811
61	.0510	.0914	.1648	86	.0151	.0343	.0787
62	.0485	.0879	.1600	87	.0143	.0330	.0764
63	.0462	.0845	.1553	88	.0136	.0317	.0742
64	.0440	.0813	.1508	89	.0130	.0305	.0720
65	.0419	.0781	.1464	90	.0124	.0293	.0699

A TABLE *ſhewing the preſent value of an annuity of one pound for any number of years, not exceeding* 90, *when intereſt is at* 5, 4, *and* 3 per cent.

Years.	Value at 5 *per cent.*	Value at 4 *per cent.*	Value at 3 *per cent.*	Years.	Value at 5 *per cent.*	Value at 4 *per cent.*	Value at 3 *per cent.*
1	0.952	0.961	0.971	21	12.821	14.029	15.415
2	1.859	1.886	1.913	22	13.163	14.451	15.939
3	2.723	2.775	2.829	23	13.488	14.857	16.444
4	3.546	3.630	3.717	24	13.799	15.247	16.936
5	4.329	4.452	4.580	25	14.094	15.622	17.413
6	5.076	5.242	5.497	26	14.375	15.983	17.877
7	5.786	6.002	6.230	27	14.643	16.329	18.327
8	6.463	6.733	7.020	28	14.898	16.663	18.764
9	7.108	7.435	7.786	29	15.141	16.984	19.188
10	7.721	8.111	8.530	30	15.372	17.292	19.600
11	8.306	8.760	9.253	31	15.593	17.588	20.000
12	8.863	9.385	9.954	32	15.803	17.873	20.389
13	9.393	9.985	10.635	33	16.002	18.148	20.766
14	9.899	10.563	11.296	34	16.193	18.411	21.132
15	10.380	11.118	11.938	35	16.374	18.665	21.487
16	10.838	11.652	12.561	36	16.547	18.908	21.832
17	11.274	12.166	13.166	37	16.711	19.142	22.167
18	11.689	12.659	13.753	38	16.868	19.368	22.492
19	12.085	13.134	14.324	39	17.017	19.584	22.808
20	12.462	13.590	14.877	40	17.159	19.793	23.115

Years.	Value at 5 *per cent.*	Value at 4 *per cent.*	Value at 3 *per cent.*	Years.	Value at 5 *per cent.*	Value at 4 *per cent.*	Value at 3 *per cent.*
41	17.294	19.993	23.412	66	19.201	23.122	28.595
42	17.423	20.186	23.701	67	19.239	23.194	28.733
43	17.546	20.371	23.982	68	19.275	23.263	28.867
44	17.663	20.549	24.254	69	19.310	23.330	28.997
45	17.774	20.720	24.519	70	19.343	23.394	29.123
46	17.880	20.885	24.775	71	19.374	23.456	29.246
47	17.981	21.043	25.025	72	19.404	23.516	29.365
48	18.077	21.195	25.267	73	19.432	23.573	29.481
49	18.169	21.341	25.502	74	19.459	23.628	29.593
50	18.256	21.482	25.730	75	19.485	23.680	29.702
51	18.339	21.617	25.951	76	19.509	23.731	29.808
52	18.418	21.747	26.166	77	19.533	23.780	29.910
53	18.493	21.873	26.375	78	19.555	23.827	30.010
54	18.565	21.993	26.578	79	19.576	23.872	30.108
55	18.633	22.109	26.774	80	19.596	23.915	30.201
56	18.698	22.220	26.965	81	19.615	23.957	30.292
57	18.760	22.327	27.151	82	19.634	23.997	30.381
58	18.819	22.430	27.331	83	19.652	24.036	30.467
59	18.876	22.528	27.506	84	19.668	24.073	30.550
60	18.929	22.623	27.676	85	19.684	24.108	30.631
61	18.980	22.715	27.840	86	19.699	24.143	30.710
62	19.029	22.803	28.000	87	19.713	24.176	30.786
63	19.075	22.887	28.156	88	19.727	24.207	30.860
64	19.119	22.968	28.306	89	19.740	24.238	30.932
65	19.161	23.046	28.453	90	19.752	24.267	31.002

A TABLE *ſhewing the difference of values of an annuity for life, at the rates of* 3 ½, *and* 4 ½ per cent. *intereſt.*

Value of an Annuity at 4 *per cent.*	Diff. of values at 4 and 4 ½ *per cent.*	Diff. of values at 4 and 3 ½ *per cent.*
6	0.2	0.2
7	0.2	0.2
8	0.3	0.3
9	0.4	0.5
10	0.5	0.6
11	0.6	0.7
12	0.7	0.8
13	0.8	0.9
14	0.9	1.0
15	1.0	1.2
16	1.1	1.3

This table may be uſed as a ſupplement to that in page 38, *&c.* thus; Suppoſe it were required to find the value of a life of forty-four, at the rate of 4 ½ *per cent.* intereſt; then looking in tab. I. page 39, againſt 44 years, under 4 *per cent.* you will find 11.0, with this enter the firſt column of the laſt table, and againſt it, in the ſecond column, you will have 0.6, which ſubtracted from 11.0, leaves 10.4 for the value required.

A METHOD *for inveſtigating the values of annuities upon lives, by approximation, without the help of tables.*

Becauſe there may be occaſion ſometimes to know the values of lives computed at different rates of intereſt, from any exhibited in the foregoing tables, and as the general method for this purpoſe, laid down in the former part of this book, is too tedious for common practice, I have endeavoured to remove that inconveniency by the help of ſome eaſy approximations. To effect this, I found it neceſſary to conſider the values of lives in two different caſes, one when the given age is leſs than 45 years, and the other when it is greater.

RULE I.

To find the value of an aſſigned life, whoſe age is not leſs than 45 years.

Multiply the difference between the given age and 92 years, by the intereſt of 1 *l.* for one year, add 2.47 to the product, and divide the ſaid difference by the product ſo increaſed, then the quotient will be the anſwer.

Note. This rule comes ſo near the truth, that the error for the general part does not amount to $\frac{1}{10}$, and ſcarce ever to $\frac{2}{10}$ of a year's purchaſe.

EXAMPLE. I.

Let the propoſed age be fifty years, and intereſt at 10 *per cent.* then ſubtracting 50 from 92, we have 42, which multiply'd by 0.1, the intereſt of 1 *l.* will be 4.2, and this added to 2.47, gives 6.67; by which divide 42, and there will come out 6.3 for the value required.

EXAMPLE. II.

Suppoſe the given age to be 55 years, and intereſt at 4 *per cent.* Here the difference between the given age and 92 years, will be 37, and the intereſt of 1 *l.* equal to 0.04; with which proceed as in the laſt example, and the required value will be found 9.36; but if the rate of intereſt had been 4 ½ *per cent.* the value then had been only 8.94; the like of any other.

I would willingly have given a rule ſimilar to the foregoing, for the other caſe where the given age is leſs than 45, but have not been able to find out any thing either ſo general or ſo ſimple, as that approximation for older lives; however, if the rate of intereſt be not leſs than 3, nor greater than 10 *per cent.* nor the age propounded, leſs than 12 years, the value of ſuch a life may be always had by the following rule, to a degree of exactneſs equal to the former.

RULE II.

To find the value of any aſſigned life, whoſe age is neither leſs than 12, *nor greater than* 45 *years.*

Take the difference between the given age and 29 years, alſo between the ſame age and 100 years, and let the former of thoſe differences be multiply'd by 10 times the intereſt of 1 *l.* for one year, and the product added to, or ſubtracted from the latter, according as the given age is greater or leſs than 29 years, and let the ſum or remainder be multiply'd by 40, and the product be reſerved: add 100 years to the given age, and multiply the ſum by 22 times the intereſt of 1 *l.* for one year; add 100 to the product, and divide the reſerved product by this ſum, and the quotient will be the value required.

EXAMPLE.

Let it be propoſed to find the value of a life of twenty, computed at the rate of 5 *per cent.*

Here the firſt mentioned difference will be 9, and the other 80, and therefore the former multiply'd by 0.5, or 10 times the intereſt of 1 *l.* will be 4.5, which ſubtracted from 80, leaves 75.5, and this multiply'd by 40, will give 3020, for the product to be reſerved. Moreover the given age, increaſed by 100 years, is 120, which multiply'd by 1.1, (=22 × .05) will be 132; therefore dividing 3020 by 232 (=100 + 132) we have 13.1 for the number of years purchaſe required.

It will be needleſs (I preſume) to offer any thing further by way of example; but, for the reader's ſatisfaction, and to remove any doubt that he may entertain with regard to the exactneſs of the above methods of ſolution, I have thought proper to add the ſubſequent table; which ſhews the values of lives to every 10th year of age, both according to thoſe methods, and alſo according to the table of Obſervations.

Age.	Value at 5 *per ct.* by the rules.	Value at 5 *per cent.* accord. to obſ.	Value at 4 *per ct.* by the rules.	Value at 4 *per cent.* accord. to obſ.	Value at 3 *per ct.* by the rules.	Value at 3 *per cent.* accord. to obſ.
20	13.01	13.0	14.86	14.8	17.25	17.2
30	11.60	11.6	13.13	13.1	15.11	15.0
40	10.31	10.3	11.53	11.5	13.14	13.2
50	9.19	9.2	10.12	10.1	11.26	11.4
60	7.86	7.9	8.53	8.4	9.36	9.2
70	6.13	6.2	6.56	6.5	7.02	6.9

The two foregoing rules are accommodated to the ſame obſervations as the preceding tables, as the beſt, undoubtedly, for the city of *London*, and parts adjacent; but if any one be deſirous of an approximation according to the *Breſlau* Obſervations, the following rule, in any caſe where the propoſed age is not leſs than 30 years, nor the rate of intereſt leſs than 3, nor greater than 6 *per cent.* will give the true anſwer very nearly. The rule is this,

Multiply the difference between the given age and 87 years, by $\frac{8}{10}$ of the intereſt of 1 *l.* for one year, and add 1.9 to the product, and divide the ſaid difference made leſs by 2, by the product ſo increaſed, and then the reſult will be the value required.

Having now ſhewn how to approximate the values of ſingle lives, it remains next to lay down ſomething in relation to annuities upon two or three lives.

RULE III.

To find the value of two lives A *and* B.

Multiply the value of the youngeſt life A by the intereſt of 1 *l.* for one year, and ſubtract the product, and alſo half the product, each from unity, dividing the laſt remainder by the former; multiply the value of the life A by the quotient thus ariſing, and divide $\frac{4}{10}$ of the ſquare of the value of the oldeſt life B, by this product, then the quotient, added to the value of the life A, will give the required value of the longeſt of the two lives.

EXAMPLE.

Suppoſe the value of the youngeſt life A equal to 12 years purchaſe, and that of the oldeſt B, equal to 10 years purchaſe, reckoning intereſt of money at 4 *per cent.* then the value of the life A multiply'd by 0.04, the intereſt of 1 *l.* for one year, will be 0.48, and half thereof equal to 0.24; theſe ſeverally taken from unity, leave 0.52 and 0.76, the former of which divided by the latter, gives 1.46, and this multiply'd by the value of the life A, will be 17.52; therefore by dividing $\frac{4}{10}$ of the ſquare of the value of the life B, by 17.52, we have 2.3, which added to 12, gives 14.3 for the value required.

RULE IV.

To find the value of the three lives A, B, C.

Firſt, find the exceſs of the value of the two youngeſt lives A, B, above that of the youngeſt life A, by Rule III. then divide the value of the oldeſt life C, by the value of the life B, and cube the quotient, and multiply that cube by half the ſaid exceſs; then the product added to the value of the two lives A and B, will be the value required.

EXAMPLE.

Let the value of the life A be ſuppoſed equal to 12 years purchaſe, that of B equal to 10 years purchaſe, and that of C equal to 8 years purchaſe, and intereſt at 4 *per cent.* then the exceſs of the value of the two lives A, B, above that of the youngeſt life A will be 2.3 (by Rule III.) moreover, the value of the life C being divided by that of B, we have 0.8, the cube whereof is 0.512, which multiply'd by 1.15, half the aboveſaid exceſs, gives 0.588, or 0.6 nearly, and this added to 14.3, the value of the two lives A, B, gives 14.9, for the value of all the three lives.

Note. The two laſt rules will ſerve indifferently, either according to the *London*, or according to the *Breſlau* Obſervations, the error, in either caſe, ſeldom exceeding $\frac{1}{8}$ of a year's purchaſe, as I have found by many repeated trials.

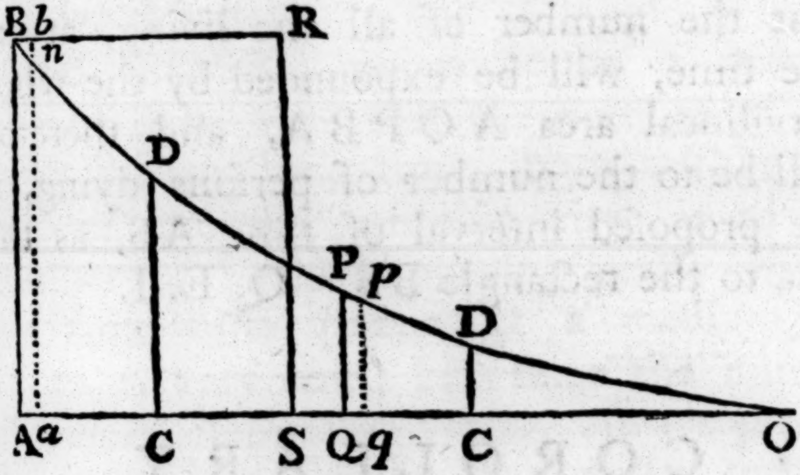
I ſhall conclude this little tract with the Solution of the following Problem, which tho' it relates not immediately to the ſubject of annuities, depends nevertheleſs upon the ſame principles.

PROBLEM XXXI.

To determine from a table of Obſervations, and the bills of mortality of any place, the number of ſouls contained in that place.

SOLUTION.

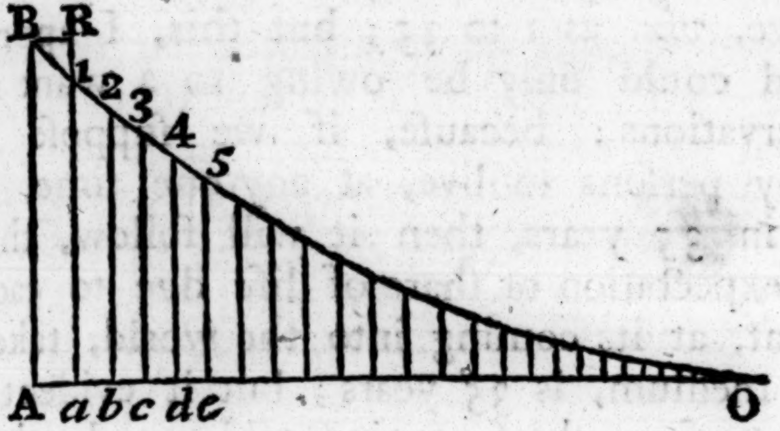
Let AO repreſent the utmoſt extent of life continued from the time of birth at A, to the extremity of old age at O, and let BDPO be a curve, whoſe ordinates CD, QP, *&c.* are, every where, to one another, as the number of perſons (found in the table) that arrive to the correſponding ages A C, A Q, *&c.* let *a b* be taken very near, nd parallel to A B, and *q p* at the ſame iſtance from Q P, and let the rectangle B S epreſent the number of perſons which die n the propoſed place, in any given interval of time A S, as found from the bills of mortality; then will the number of perſons that



ome into, or go out of, being in the interval A *a*, be repreſented by the rectangle B *a*; nd conſequently the number of all the living, whoſe ages are comprized in this interval, by the area B A *a n* B, or the ſame ectangle B *a* very near: Wherefore, ſince he number of individuals, whoſe ages now ake up A *a*, will in the time A Q be educed in the ratio of A B to P Q, ſo as o be expounded by the area P Q *q p* P, the umber of all the living at the end of that time, whoſe ages are then included between Q and q, will be repreſented by the area $PQqpP$; ſince it is evident that the ages of all who come into being, before or after thoſe now included between A and a, will then be either greater than Aq, or leſs than AQ. Therefore as the number of the living, at all equal ages, is ſuppoſed to continue conſtantly the ſame, and as the above reaſoning holds every where throughout the whole extent of life AO, it is manifeſt that the number of all the living, at any one time, will be expounded by the whole curvilineal area $AOPBA$, and therefore will be to the number of perſons dying, in the propoſed interval of time AS, as that area to the rectangle BS. Q. E. I.

COROLLARY.

Hence if the whole term of life AO be divided into ſeveral ſmall intervals Aa, ab, bc, *&c.* ſo that each interval may repreſent one whole year, and ordinates AB, $a1$, $b2$, *&c.* be deſcribed at the points of diviſion, and the firſt AB be taken 1280, then will $a1$, $b2$, $c3$, *&c.* according to the foregoing table of Obſervations, taken from the bills of mortality of the city of *London*, be 870, 700, 635, 600, *&c.* reſpectively; from whence the area $ABOA$ will



will come out 25500 very near, which area is, therefore, to the rectangle B *a*, as 25500 to 1280, or as 20 to 1 nearly; whence it appears that the number of the living, at any one time, born within the bills of mortality of this city, is to the number of births happening yearly within the ſame bills (taken at a medium) as 20 to 1, very near; but ſince the number of burials happening yearly, always exceeds the number of births, by reaſon (as has been before obſerved) of the continual afflux of people from all parts to town, the proportion of that number to the whole body of the inhabitants, will be conſiderably different from the proportion above given; yet may be nearly eſtimated by comparing together the number of burials and chriſtenings, *&c.*

And (from a method too tedious to be inſerted here) I make it as 1 to 26 very near. I know indeed that a certain author, conſiderable in theſe kind of diſquiſitions, gives the proportion very different from that above, *viz.* as 1 to 35; but this, I apprehend could only be owing to a want of obſervations; becauſe, if we ſuppoſe as many perſons to live, at any one time, as die in 35 years, then it will follow, that the expectation or ſhare of life due to each infant, at its coming into the world, taken at a medium, is 35 years; but it evidently appears from the foregoing table, and other undoubted obſervations, that there is ſcarce any part of life wherein the expectation is ſo great as 35 years, much leſs in the very beginning of it, attended by ſo many caſualties and dangers.

APPENDIX,

CONTAINING

SOME REMARKS *on* MR. DEMOIVRE'S BOOK *on the ſame Subject, with* ANSWERS *to ſome* PERSONAL *and* MALIGNANT MISREPRESENTATIONS, *in the* PREFACE *thereof.*

IT is not without ſome concern, that I think myſelf obliged to take notice of the treatment of an author, of whoſe judgment and candour I had conceived a very high opinion, and whoſe name and writings I have always mentioned with reſpect. It is not my deſign to expatiate on the unſeemlineſs of this gentleman's uſage, nor to gratify a paſſion, which inſinuations ſo groſs muſt naturally excite in a mind that looks with contempt on ſuch unfair proceedings; but only to offer a few particulars to the conſideration of the public, with no other view than to clear myſelf from a charge ſo highly injurious, and do juſtice to the foregoing work. In order to which it will be requiſite to take a ſhort view of what both of us have done in the ſubject.

The Rules whereby the various queſtions in annuities, *&c.* are reſolved, are, not improperly, reduced under three heads or claſſes; in the firſt of which may be included thoſe that relate to ſingle lives; in the ſecond, thoſe that have regard to ſucceſſive lives and copyholds; and in the third, ſuch as ſhew the values of joint lives and reverſions.

As to what comes under the firſt of theſe heads, it muſt be granted, that this author has ſucceeded extremely well, what he has done therein being both curious and very exact, according to the obſervations whereon he builds, and the hypotheſis he has taken ſo much pains to eſtabliſh: but, tho' his method is curious, and his rule exact, does it any where appear that I have made free with the one or the other? have I followed his hypotheſis? is not my method quite different, and founded upon different obſervations? it is true, I have given practical rules for determining the values of ſingle lives, both according to the *London* and *Breſlaw* obſervations; but theſe, every one will ſee, have no ſort of dependence on, or connexion with his. Nevertheleſs I can venture to affirm, that mine have ſo far the advantage, that they are not only more ſimple, as requiring the uſe of no ſort of tables, but bring out the concluſions, for the general part, rather more exact than his. As an inſtance hereof, let it be propoſed to find the value of a life of ten years of age, at an intereſt of 6 *per Cent*. Firſt, according to this gentleman's method, let the given age be ſubtracted from 86, and there remains 76; then the value of an annuity *certain* for the term of 76 years muſt be found, which will be 16.467; this multiply'd by 1.06, the rate of intereſt, gives 17.455; which being divided by 76, we have 0.22967; this ſubtracted from unity, leaves 0.77033; which divided by .06 quotes 12.84 for the value ſought. But, according to my method, (as now improved in p. 5 of the preface) the given age being taken from 87, the remainder will be 77: this multiply'd by .048, or $\frac{8}{10}$ of the intereſt of 1*l.* gives 3.696; to which adding 1.9 we have 5.596; by which divide 75, and the quotient 13.40 will be the anſwer: differing from, 13.44, the true value given by Dr. *Halley*, from the table of obſervations, by only 0.04. But, as a farther confirmation of what is above advanced, and to ſhew that it is not only in one particular caſe, that the concluſion comes out ſo near, I have, in the annexed Table, exhibited the values of an-

10	13.40	12.84	13.44
20	12.70	12.30	12.78
30	11.61	11.61	11.72
40	10.57	10.70	10.57
50	9.26	9.49	9.21
60	7.58	7.83	7.60
70	5.32	5.50	5.32
80	2.14	2.18	

nuities for every tenth year: Where it is to be obſerved, that the ſecond and third columns expreſs the values that reſult from mine, and that gentleman's rules, and the fourth, the true value, as given by Dr. *Halley*. It is alſo to be obſerved that the values above given, are adapted to the *Breſlaw* obſervations, not becauſe they appear the moſt proper foundation to build upon *in this place*, but that I might compare my concluſions with his, and thereby ſhew that the method I have followed is capable of being applied, with equal advantage, to any table of obſervations.

Next, as to what falls under the ſecond of the forementioned heads, are not his rules for ſucceſſive lives, in his former edition, (which was the only edition extant, at the publication of the preceding work) delivered without any kind of demonſtration? do they not always bring out the concluſions too large? do not they differ from thoſe in his new edition? and are not the rules which I have given, general and clearly inveſtigated? how then can I have *mutilated his propoſitions?* Are not my demonſtrations more ſimple than thoſe in his new edition, and is not the principle whereon they are founded alſo more ſimple and evident? how then can I have *obſcured his demonſtrations?* are not his rules for renewing of leaſes, or replacing of lives, both in his old and new edition, confined to that particular caſe where all the lives are equal? (which can ſcarce ever happen in practice.) Do not my rules extend to lives of all ages, however unequal, and yet are they not more ſimple and eaſy in practice than his? how then can I have *worked by his rules?* Laſtly, is not all that he has advanced, on this head, from *p.* 29 to *p.* 33, and, again from *p.* 101 to *p.* 103, as a matter of difficulty and curious ſpeculation, a particular caſe only of what I had before given in the ſolution of my third problem? I have repreſented above, that the principle whereon my demonſtrations are grounded, (explained in *p.* 76 and 89) is more clear and ſimple than his: it may not be improper here to give an inſtance of it. Let there be ſuppoſed an annuity, depending upon any number of lives, A, B. C, D, *&c.* ſo as to continue 'till the failing of one, two, three, or any other number, of thoſe lives; and let it be further ſuppoſed that upon the failing of theſe lives, Q, or his heirs, are to receive any given ſum, annuity or reverſion, whoſe value is denoted by s: then in order to determine the preſent value of Q's expectation, let the value of the annuity, under the forementioned contingency, (which is given by my third problem) be repreſented by p, and let the perpetuity be denoted by e. Now ſince the value of the reverſion for ever, after the failing of the propoſed number of lives, to be received as ſoon as thoſe lives are extinct, is the perpetuity, it will be to the value s, to be received at the ſame time, as e to s; therefore the preſent value of the reverſion, muſt be to the preſent value of the ſum s, in the ſame ratio of e to s; but the preſent value of the reverſion is given equal to $e-p$; wherefore it will be, as $e : s :: e-p : \frac{\overline{e-p}\times s}{e}$, the true value of Q's expectation. But if after the deceaſe of the propoſed number of lives, others were to be nominated, to fill up their place; and after the deceaſe of theſe, others, and ſo on for ever, and Q and his heirs were to receive the like ſum at each replacement: Then let the value of the lives, in the leaſe, under the forementioned limitations, as they ſtand after the ſecond, third, fourth, *&c.* replacements, be denoted by q, r, s, t, &c. reſpectively; and from the very ſame way of reaſoning, it is evident, that the value of the ſum to be received at the ſecond replacement will be to the value of the ſame ſum, to be received at the time of the firſt replacement, as e to $e-q$; therefore the value that might be received, as an equivalent, at the time of the firſt replacement is $\frac{\overline{e-q}\times s}{e}$: But by what is already ſhewn it will be, as $e : e-p$: ſo is this laſt value $\frac{\overline{e-q}\times s}{e}$ to $\frac{e-p}{e}\times\frac{e-q}{e}\times s$, to its equivalent in preſent money. After the ſame manner, the preſent value of the ſum to be received at the third replacement, will be found $\frac{e-p}{e}\times\frac{e-q}{e}\times\frac{e-r}{e}\times s$; whence the law of continuation is manifeſt. Now let any one compare this with what our author has delivered upon this head from *p.* 97 to *p.* 103, and then judge whether what I have advanced be not juſt, and whether all that he has done, either with regard to copyholds, renewing of leaſes, or perpetual advowſons, be not included in what is above laid down, beſides the ſolutions to his fourteenth and twenty-ſixth problems. But if we deſcend to that particular caſe, where all the values p, q, r, &c. are equal, (which is the only caſe that he has conſidered) then the preſent value of the ſum to be received at the ſecond replacement will be expreſſed by $\frac{\overline{e-p}^{2}}{e^{2}}\times s$; that, at the third, by $\frac{\overline{e-p}^{3}}{e^{3}}\times s$ &c. Whence it is manifeſt that the preſent value of all the ſums which Q and his heirs may receive for ever, will be truly defined by the following geometrical progreſſion, $\frac{e-p\times s}{e}+\frac{\overline{e-p}^{2}\times s}{e^{2}}+\frac{\overline{e-p}^{3}\times s}{e^{3}}$, &c. or its equal $\frac{\overline{e-p}\times s}{p}$. It would be needleſs to inſiſt further on this matter here, as I have conſidered it at large in the book, where it is illuſtrated by proper examples. I ſhall therefore now proceed to the third head, relating to the values of joint lives and reverſions.

Here our author ſets out with a new hypotheſis, which is directly contrary to experience; in conſequence whereof his concluſions are rendered uncertain, and liable to very conſiderable errors. I am ſenſible that no ſmall pains have been taken, both in his old and new edition, to bring the world into a belief, that the concluſions, thence derived, would be always very near the truth; nay, in confirmation thereof, we have an example given, where the error is inconſiderable.

But all this will not ſhew his rules to be what he would repreſent them; much leſs that I could meanly counterfeit and diſguiſe them. No, I muſt beg leave to aſſure the author I had not that high opinion of them: I ſaw how far they could be rely'd on. It may not be improper, however, to give here a few inſtances of their inaccuracy. Firſt, let it be required to find the value of three equal *joint* lives, at an intereſt of 4 *per cent.* each life ſingly taken being worth fourteen years purchaſe; then, according to his rule the anſwer will come out 7.3; which, granting his firſt hypotheſis to be true (as it is very nearly) is leſs than the truth, by 1.2, or more than a whole year's purchaſe. Again, ſuppoſing it were required to find the value of an annuity for the longeſt of thoſe three equal lives; then, by his rule, the reſult will be 20.4, exceeding the true value 18.7, (according to his firſt hypotheſis) by almoſt two years purchaſe. This difference when the value is large, may not ſeem ſo conſiderable, but in reverſions, and the putting in of new lives, the error is the ſame, and in many caſes amounts to more than $\frac{1}{4}$ of the whole value: Thus, if A holds an annuity upon a ſingle life, worth fourteen years purchaſe, and would put in two other lives, of the ſame age, to hold it as long as any one of the three is in being; then the ſum that he ought, in juſtice, to pay for this conſideration will be 4.7; whereas, according to our author's rules, the anſwer will come out no leſs than 6.4. In like manner, if the values of the lives were unequal, ſuppoſe that of B 8, and that of C 10 years purchaſe; then the value of the reverſion of the life B, after the life C, will come out 2.73; which is too great by $\frac{1}{3}$ of the whole value.

'Tis true there are other rules, in his new edition, from which the values of joint lives and reverſions are determined, and which he aſſures his reader were *originally derived from the former*, and that, *whether one, or the other (method) is uſed, the concluſions will* *very little differ*; only, *that the firſt is better adapted to annuities paid in money, and the other to ſuch as are paid by a grant of lands.* As to what is advanced concerning the nearneſs of the concluſions, it is indeed very juſt, the difference ſcarce ever exceeding $\frac{2}{10}$ of a year's purchaſe; but muſt it not ſeem a little ſtrange that he ſhould here make ſuch an extraordinary *ſhew* of exactneſs, and appear ſo ſolicitous about a ſmall difference of $\frac{1}{8}$ or $\frac{1}{10}$ of a year's purchaſe, when at the ſame time, both his methods differ from the truth by more than five times as much?

The errors hitherto taken notice of, are only ſuch as naturally ariſe from his building upon an ill-grounded hypotheſis; but who would not wonder if this gentleman, who every where diſcovers an air of ſelf-ſufficiency, who will allow of no merit but his own, and who treats with ſuch ſuperlative contempt thoſe that have dared to write upon the ſame ſubject after him; I ſay, who would not wonder, if this gentleman has condeſcended to give the world rules which are abſurd, and propoſitions that are abſolutely falſe, even, according to his own principles?

The firſt, I ſhall take notice of, which I apprehend to be of this number, is in page 52 of his former edition: The queſtion is to find the value of the reverſion of one life, after two joint lives; and his rule is as follows. *From the value of the life in expec-*

tation, ſubtract the value of the two joint lives in poſſeſſion, there remains the value of the reverſion. Now the reader will eaſily ſee, that this rule cannot be true; becauſe the value of the two joint lives in poſſeſſion, if they be young lives, may exceed the value of the life in expectation, by ſeveral years purchaſe, in which caſe the value of the reverſion will come out a great deal leſs than nothing, which is manifeſtly abſurd. The true method of ſolving this problem, I have given and demonſtrated *(without a crowd of uſeleſs ſymbols)* in my fifth problem, of which it is only a particular caſe.

The rule in p. 55, of the ſame edition, I alſo reckon to be *in the number of the* falſe *ones*; but as this is corrected in the new edition, I ſhall take no further notice of it here, but only obſerve that as it now ſtands, it is the very ſame with what I had given and demonſtrated at large, in p. 74.

The rules given (without demonſtration) for ſucceſſive lives, in his firſt edition, tho' not ſo erroneous as the preceding, are not true, nor the ſame with thoſe in his new edition; which laſt are right, and bring out the very ſame concluſions, as thoſe, which I have demonſtrated in p. 89, 90, 91, tho' expreſs'd in a different manner.

Beſides theſe errors, in his former edition, there are others in the new, as conſiderable. His twenty-third problem will afford us one inſtance of it. The terms in which he propoſes that problem are theſe; A *who is thirty years of age, buys an annuity for a limited time of his life, ſuppoſe ten years, on condition that if he dies before the expiration of that time, the purchaſe-money is wholly to be loſt to his heirs; to find the preſent value of the purchaſe, ſuppoſing intereſt at* 5 *per cent.*

Now the true, and moſt eaſy way of reſolving this problem, (ſuppoſing the values of ſingle lives to be given, from the tables) I take to be as follows.

1° Since A can enjoy the annuity, no longer than ten years, it is evident that the preſent value of what he may happen to receive, from time to time, during that term will be equal to the preſent value of a life of thirty, decreaſed by the preſent value of all the rents or ſums he would have a chance of receiving after the age of forty, was the annuity granted for the whole term of life.

2° Now the preſent value of all the rents that A would have a chance of receiving after the age of forty, if there was a certainty of his living to that age, would be the value of a life of forty, diſcounted for ten years, at the rate of 5 *per cent.*

3° But ſince there is a probability that he never attains to the age of forty, it is manifeſt that the value ſo diſcounted, ought to be further diminiſhed according to the meaſure of that probability; and then, this aſt ſum being ſubtracted from the preſent value of a life of thirty, the remainder will be the value required: Or, univerſally let the number of years on which the annuity is granted be $=n$, the probability of A's living to the end of thoſe years $=p$, the preſent value of 1*l.* to be received n years hence $=m$, the value of the life $A=M$, and the value of a life n years older$=P$; then will $M-mpP$ expreſs the true value of the purchaſe. Now in the caſe above, n being $=10$, the values of m, p, M and P, will according to this author's own tables, be equal to $\frac{445}{531}$, 0.6139, 12.99 and 11.83 reſpectively, and therefore 6.904 the true value of the purchaſe; but by his rule, the value comes out only 5.73, being leſs than the truth, by 1.174. But the error here is but ſmall, when compared with what it is in ſome other caſes: for if it were required to find the value of an annuity, for the firſt ten years of a life of one year old; then by his rule the anſwer would come out only 2.013, whereas the true value is no leſs than 5.953, which is almoſt three times as great, as he makes it.

His ſolution to his twenty-fifth problem, ſeems to be of the ſame kind, with that above mentioned; the problem is this. A *has an annuity of* 100l. *to continue thirty-*

one years; A *whoſe life is worth fourteen years purchaſe, ſuppoſing intereſt at* 4 per cent. *is to enjoy an annuity of* 100l. *during the term of thirty-one years*; B *has the reverſion of it after the deceaſe of* A, *for the term remaining*; *to find the value of* B's *expectation*. This queſtion indeed ſeems a little oddly, as well as obſcurely expreſſed, but I think it can bear no other meaning than this, that A, who has the right of an annuity for the term of thirty-one years *certain*, impowers B and his heirs to receive or hold the annuity after his deceaſe, (if he ſhould happen to die before the end of thirty-one years) until the remainder of the term is expired. This being ſuppoſed, the method of ſolution may be as follows: firſt, let the value of the propoſed annuity, for the limited term of thirty-one years, upon the contingency of A's not living to the end of that term, be found; the manner of doing which is already explained; then let the value ſo found, be ſubtracted from the value of an annuity *certain* for thirty-one years, and the remainder, it is manifeſt, will be the true value of B's expectation: Thus, if the age of A be thirty-five years, (anſwering in our author's table, to fourteen years purchaſe) and the annuity be 100 *l.* (as above propoſed) then the preſent value of all the ſums, that A may happen to receive from time to time, during the term of thirty-one years, will be found to be 1319 *l.* which taken from 1758 *l.* the value of an annuity *certain* for thirty-one years, there remains 439 *l.* for the true value of B's expectation; which this author makes 773 *l.* exceeding the truth by no leſs than 334 *l.* or nearly half the whole value. As the numbers, from whence theſe concluſions are derived, are taken from this gentleman's own tables, it cannot be ſuppoſed that I have been guilty of any injuſtice in that reſpect, but that the rules themſelves are actually falſe according to his own principles. Indeed it might be wiſhed that he had given his demonſtrations along with the rules, and then it would have been poſſible to point out wherein he was miſtaken; but as he has not done it, it may ſuffice to have ſhewn that the rules cannot be true, by the conſequences they lead to.

Theſe are ſome of the errors in this author's book, which I have taken notice of, to ſhew the world, that notwithſtanding he had done *ſome* things on the ſubject very curious, there was ſtill room left for improvements: Had not I thought ſo, I can aſſure my reader, I ſhould not have undertaken the foregoing work, (however mean an opinion, this author may perſuade the world to entertain of it) nay, if he had not thought ſo himſelf, how comes it to paſs that in his new *plainer and more correct edition*, almoſt every thing is done in a quite different manner? This, then, being the caſe, why ſhould this author moleſt himſelf, about my right of offering to public conſideration what I humbly conceived to be real improvements on the ſubject, unleſs he thought no one capable of it but himſelf? In ſhort had I been really guilty of mutilating his propoſitions, as he would inſinuate, he muſt allow, at leaſt, that I had ſo much REGARD, (in his ſenſe of the word) as to diſtinguiſh between the good and the bad. I have no propoſitions that are falſe.

Laſtly, I appeal to all mankind, whether in his treatment of me, he has not diſcovered an air of ſelf-ſufficiency, ill-nature, and inveteracy, unbecoming a gentleman.

FINIS.